HOMEWORK #13 (Chapter 5 Series solutions of Linear Equations)

In problem 1, find the solution by using the Taylar series method and the recurrence relations method.

1. $y'' + e^x y' - y = 0$ (Exercises 5.1 Problem 30)

Ans: **Taylar series method**

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots$$

Let $y(0) = a$, $y'(0) = b$

$y''(0) = a - b$, $y'''(0) = -a + b$, $y^{(4)}(0) = -b$

$$\therefore y(x) = a + \frac{b}{1!}x + \frac{(a-b)}{2!}x^2 - \frac{(a-b)}{3!}x^3 - \frac{b}{4!}x^4 + \cdots$$

$$= a(1 + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots) + b(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots)$$

recurrence relations method

Let $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots) - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + \cdots) + [c_1 + (c_1 + 2c_2)x + (\frac{c_1}{2} + 2c_2 + 3c_3)x^2 + (4c_4 + 3c_3 + c_2 + \frac{c_1}{6})x^3 + \cdots]$$

$$-(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots) = 0$$

$$2c_2 + c_1 - c_0 = 0, \quad 6c_3 + (c_1 + 2c_2) - c_1 = 0, \quad 12c_4 + (\frac{c_1}{2} + 2c_2 + 3c_3) - c_2 = 0$$

$$\therefore c_2 = \frac{c_0 - c_1}{2}, \quad c_3 = \frac{-c_2}{3} = \frac{-c_0 + c_1}{6}, \quad c_4 = -\frac{(\frac{c_1}{2} + c_2 + 3c_3)}{12} = \frac{-c_1}{24}$$

$$\because y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots = c_0 + c_1 x + \frac{c_0 - c_1}{2}x^2 + \frac{-c_0 + c_1}{6}x^3 + \frac{-c_1}{24}x^4 + \cdots$$

Choosing $c_0 = 1$, $c_1 = 0 \to y_1 = 1 + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$

Choosing $c_0 = 0$, $c_1 = 1 \to y_2 = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \cdots$

$$y = \overline{c_1} y_1 + \overline{c_2} y_2 = \overline{c_1}(1 + \frac{x^2}{2} - \frac{x^3}{6} + \cdots) + \overline{c_2}(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \cdots)$$

In problems 2~3, $x = 0$ is a regular singular point of the given differential equation. Use the method of Frobenius to obtain at least one series solution about $x=0$. Use $y_2(x)=y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$ where necessary and a CAS, if instructed, to find a second solution. From the general solution on $(0,\infty)$.

2. $xy''-xy'+y=0$ (Exercises 5.2 Problem 27)

Ans: Let $y=\sum_{n=0}^{\infty}c_nx^{n+r}$, $y'=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}$, $y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}-\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}+\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$\sum_{n=-1}^{\infty}(n+r+1)(n+r)c_{n+1}x^{n+r}-\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}+\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$x^r\{r(r-1)c_0+[\sum_{n=0}^{\infty}(n+r+1)(n+r)c_{n+1}-\sum_{n=0}^{\infty}(n+r-1)c_n]x^n\}=0$$

$$r(r-1)=0\rightarrow r=1,0 \quad \text{and} \quad c_{k+1}=\frac{(k+r-1)c_k}{(k+r+1)(k+r)},\quad k=0,1,2\cdots$$

$$r=1\rightarrow c_{k+1}=\frac{kc_k}{(k+2)(k+1)},\quad c_1=0=c_2=c_3 \qquad \therefore y_1=c_0x=x$$

Method 1:

$$y_2(x)=y_1\ln|x|+\sum_{n=0}^{\infty}c_n^*x^{n+r_2}=x\ln|x|+\sum_{n=0}^{\infty}c_n^*x^n$$

$$y_2'(x)=\ln|x|+1+\sum_{n=1}^{\infty}nc_n^*x^{n-1},\qquad y_2''(x)=\frac{1}{x}+\sum_{n=2}^{\infty}n(n-1)c_n^*x^{n-2}$$

$$x(\frac{1}{x}+\sum_{n=2}^{\infty}n(n-1)c_n^*x^{n-2})-x(\ln|x|+1+\sum_{n=1}^{\infty}nc_n^*x^{n-1})+x\ln|x|+\sum_{n=0}^{\infty}c_n^*x^n=0$$

$$1+\sum_{n=2}^{\infty}n(n-1)c_n^*x^{n-1}-x\ln|x|-x-\sum_{n=1}^{\infty}nc_n^*x^n+x\ln|x|+\sum_{n=0}^{\infty}c_n^*x^n=0$$

$$1-x+\sum_{n=1}^{\infty}n(n+1)c_{n+1}^*x^n-\sum_{n=1}^{\infty}nc_n^*x^n+\sum_{n=0}^{\infty}c_n^*x^n=0$$

$$1-x+c_0^*+\sum_{n=1}^{\infty}[n(n+1)c_{n+1}^*+(1-n)c_n^*]x^n=0$$

$$c_0^*=-1,\quad c_2^*=\frac{1}{2} \quad \text{and} \quad c_{k+1}^*=\frac{-(1-k)c_k^*}{k(k+1)},\quad k=2,3,4\cdots$$

$$\therefore c_3^*=\frac{1}{6}c_2^*=\frac{1}{12},\quad c_4^*=\frac{1}{6}c_3^*=\frac{1}{72}$$

$$y_2(x)=x\ln|x|-1+c_1^*x+\frac{1}{2}x^2+\frac{1}{12}x^3+\frac{1}{72}x^4+\cdots=x\ln|x|-1+\frac{1}{2}x^2+\frac{1}{12}x^3+\frac{1}{72}x^4+\cdots$$

Method 2:

$$y_2(x)=y_1(x)\int\frac{e^{-\int p(x)dx}}{y_1^2(x)}dx=x\int\frac{e^{\int dx}}{x^2}dx=x\int\frac{e^x}{x^2}dx=x\int\frac{1}{x^2}(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)dx$$

$$=x\int(\frac{1}{x^2}+\frac{1}{x}+\frac{1}{2!}+\frac{x}{3!}+\cdots)dx=x(\frac{-1}{x}+\ln|x|+\frac{1}{2}x+\frac{1}{12}x^2+\cdots)$$

$$=-1+x\ln|x|+\frac{1}{2}x^2+\frac{1}{12}x^3+\cdots$$

$$\therefore y=\overline{c_1}y_1+\overline{c_2}y_2$$

3. $xy''+(1-x)y'-y=0$ (Exercises 5.2 Problem 29)

Ans: Let $y=\sum_{n=0}^{\infty}c_nx^{n+r}$, $y'=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}$, $y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$

$$x[\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}]+(1-x)\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}-\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$\sum_{n=-1}^{\infty}(n+r+1)(n+r)c_{n+1}x^{n+r}+\sum_{n=-1}^{\infty}(n+r+1)c_{n+1}x^{n+r}-\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}-\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$x^r\{[r(r-1)c_0+rc_0]x^{-1}+[\sum_{n=0}^{\infty}(n+r+1)(n+r+1)c_{n+1}-(n+r+1)c_n]x^n\}=0$$

$r=0,0$ and $c_{k+1}=\frac{c_k}{(k+r+1)}$, $k=0,1,2\cdots$, $c_1=\frac{c_0}{(r+1)}$, $c_2=\frac{c_1}{(r+2)}=\frac{c_0}{(r+1)(r+2)}$

$$\therefore y=c_0x^r(1+\frac{x}{(r+1)}+\frac{x^2}{(r+1)(r+2)}+\cdots)=x^r(1+\frac{x}{(r+1)}+\frac{x^2}{(r+1)(r+2)}+\cdots)$$

$r=0$, $y_1=1+x+\frac{x^2}{2}+\cdots$

$$\frac{\partial y}{\partial r}=x^r\ln|x|(1+\frac{x}{(r+1)}+\frac{x^2}{(r+1)(r+2)}+\cdots)+x^r(\frac{-x-x^2}{(r+1)^2}+\frac{x^2}{(r+2)^2}+\cdots)$$

$r=0$, $y_2=\ln|x|(1+x+\frac{x^2}{2}+\cdots)+(-x-x^2+\frac{x^2}{4}+\cdots)$

$$\therefore y=\overline{c_1}y_1+\overline{c_2}y_2$$