HOMEWORK #14 (Chapter 8 Matrics)

In Problem 1, find the eigenvalues and eigenvectors of the given matrix.

1. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Exercises 8.8 Problem 22)

Ans: $\lambda_1 = \lambda_2 = 0 \rightarrow \{1 \quad 0 \quad 0\}^T, \{0 \quad 1 \quad 0\}^T$; $\lambda_3 = 1 \rightarrow \{0 \quad 0 \quad 1\}^T$

The eigenvalues of A^{-1} are the reciprocals of a nonsingular matrix A. Furthermore, the eigenvectors for A and A^{-1} are the same. In problem 2, verify these facts for the given matrix.

2. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$ (Exercises 8.8 Problem 24)

Ans: $\lambda_1 = 1 \rightarrow \{-1 \quad 1 \quad 2\}^T$, $\lambda_2 = 2 \rightarrow \{-2 \quad 1 \quad 4\}^T$, $\lambda_3 = 3 \rightarrow \{-1 \quad 1 \quad 4\}^T$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -6 & 2 \\ -1 & 9 & -2 \\ -4 & 12 & -2 \end{bmatrix}$$

$$\lambda_1 = 1 \rightarrow \{-1 \quad 1 \quad 2\}^T, \; \lambda_2 = \frac{1}{2} \rightarrow \{-2 \quad 1 \quad 4\}^T, \; \lambda_3 = \frac{1}{3} \rightarrow \{-1 \quad 1 \quad 4\}^T$$

A matrix A is singular if and only if $\lambda = 0$ is an eigenvalue. In Problem 3, verify that the given matrix A is singular. Find the characteristic equation for A and verify that $\lambda = 0$ is an eigenvalue.

3. $\begin{bmatrix} 6 & 0 \\ 3 & 0 \end{bmatrix}$ (Exercises 8.8 Problem 25)

Ans: $|A - \lambda I| = \lambda(\lambda - 6) = 0$, $\lambda_1 = 0$, $\lambda_2 = 6$

In Problem 4, use the procedure illustrated in Example 6 to identify the given conic section. Graph.

4. $5x^2 - 2xy + 5y^2 = 24$ (Exercises 8.12 Problem 31)

Ans: $x^T A x = [x \quad y]\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = 24$, $\lambda_1 = 6 \rightarrow \{1 \quad -1\}^T$, $\lambda_2 = 4 \rightarrow \{1 \quad 1\}^T$,

$$P=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix},\quad X=PX'$$

$$x^T Ax=(PX')^T APX'=X'^T P^T APX'=X'^T P^{-1}APX'=X'^T DX'=\begin{bmatrix}X & Y\end{bmatrix}\begin{bmatrix}6 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}=24$$

In Problem 5, Find the Jordan form of matrix A.

5. $A=\begin{bmatrix}0 & 9 & -4\\ -1 & -2 & 2\\ -2 & 0 & 2\end{bmatrix}$

Ans: $\lambda_1=-2\rightarrow\{9\quad 2\quad 9\}^T$, $\lambda_2=\lambda_3=1\rightarrow\{1\quad 1\quad 2\}^T$

$$[A-\lambda_2 I]\{x_3\}=\{x_2\}\rightarrow\{x_3\}=\{0\quad 1\quad 2\}^T$$

$$P^{-1}AP=P^{-1}(A\underset{\sim}{x_1}\quad A\underset{\sim}{x_2}\quad A\underset{\sim}{x_3})=P^{-1}(\lambda_1\underset{\sim}{x_1}\quad \lambda_2\underset{\sim}{x_2}\quad \lambda_2\underset{\sim}{x_3}+\underset{\sim}{x_2})$$

$$=(\lambda_1 P^{-1}\underset{\sim}{x_1}\quad \lambda_2 P^{-1}\underset{\sim}{x_2}\quad \lambda_2 P^{-1}\underset{\sim}{x_3}+P^{-1}\underset{\sim}{x_2})=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 1\\ 0 & 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}-2 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

In Problem 6, Find the A^{10} of matrix A.

6. $A=\begin{bmatrix}-2 & 4\\ -1 & 3\end{bmatrix}$

Ans: $\lambda_1=2\rightarrow\{1\quad 1\}^T$, $\lambda_2=-1\rightarrow\{4\quad 1\}^T$; $P=\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix}$, $P^{-1}=\begin{bmatrix}-1/3 & 4/3\\ 1/3 & -1/3\end{bmatrix}$

$$A=PDP^{-1},\quad \therefore A^{10}=PD^{10}P^{-1}=\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}2^{10} & 0\\ 0 & (-1)^{10}\end{bmatrix}\begin{bmatrix}-1/3 & 4/3\\ 1/3 & -1/3\end{bmatrix}$$

In Problem 7~9, use the method of section to compute A^m. Use this result to compute the indicated power of the matrix A.

7. $A=\begin{bmatrix}-1 & 2\\ 0 & -3\end{bmatrix}$; m=6. (Exercises 8.9 Problem 6)

Ans: $\lambda^2+4\lambda+3=0$, $\lambda_1=-3$ and $\lambda_2=-1$; $\lambda^m=c_0+c_1\lambda\rightarrow\begin{cases}(-3)^m=c_0-3c_1\\ (-1)^m=c_0-c_1\end{cases}$

$$c_0 = \frac{1}{2}[-(-3)^m + 3(-1)^m],\quad c_1 = \frac{1}{2}[-(-3)^m + (-1)^m],$$

$$A^m = c_0 I + c_1 A = \begin{bmatrix} (-1)^m & -(-3)^m + (-1)^m \\ 0 & (-3)^m \end{bmatrix},\quad \therefore A^6 = \begin{bmatrix} 1 & -728 \\ 0 & 729 \end{bmatrix}$$

8. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$; m=10. (Exercises 8.9 Problem 7)

Ans: $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$, $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$; $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$

$$\begin{cases} (-1)^m = c_0 - c_1 + c_2 \\ 1 = c_0 + c_1 + c_2 \\ 2^m = c_0 + 2c_1 + 4c_2 \end{cases} \rightarrow c_0 = \frac{1}{3}[3 + (-1)^m - 2^m], c_1 = \frac{1}{2}[1 - (-1)^m],$$

$$c_2 = \frac{1}{6}[-3 + (-1)^m + 2^{m+1}]$$

$$A^m = c_0 I + c_1 A + c_2 A^2 = \begin{bmatrix} 1 & -1 + 2^m & -1 + 2^m \\ 0 & \frac{1}{3}[(-1)^m + 2^{m+1}] & \frac{-2}{3}[(-1)^m - 2^m] \\ 0 & \frac{1}{3}[-(-1)^m + 2^m] & \frac{1}{3}[2(-1)^m + 2^m] \end{bmatrix};$$

$$\therefore A^{10} = \begin{bmatrix} 1 & 1023 & 1023 \\ 0 & 683 & 682 \\ 0 & 341 & 342 \end{bmatrix}$$

9. $A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$; m=10. (Exercises 8.9 Problem 9)

Ans: $-\lambda^3 + 3\lambda^2 + 6\lambda - 8 = 0$, $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 4$; $\lambda^m = c_0 + c_1\lambda + c_2\lambda^2$

$$\begin{cases} (-2)^m = c_0 - 2c_1 + 4c_2 \\ 1 = c_0 + c_1 + c_2 \\ 4^m = c_0 + 4c_1 + 16c_2 \end{cases} \rightarrow c_0 = \frac{1}{9}[8 + (-1)^m 2^{m+1} - 4^m],\quad c_1 = \frac{1}{18}[4 - 5(-2)^m + 4^m],$$

$$c_2 = \frac{1}{18}[-2 + (-2)^m + 4^m]$$

$$A^m = c_0 I + c_1 A + c_2 A^2 = \begin{bmatrix} \frac{1}{9}[(-2)^m + (-1)^m 2^{m+1} + 3 * 2^{2m+1}] & \frac{1}{3}[-(-2)^m + 4^m] & 0 \\ \frac{-2}{3}[(-2)^m - 4^m] & \frac{1}{3}[(-1)^m 2^{m+1} + 4^m] & 0 \\ \frac{1}{3}[-3 + (-2)^m + 2^{2m+1}] & \frac{1}{3}[-(-2)^m + 4^m] & 1 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 699392 & 349184 & 0 \\ 698368 & 350208 & 0 \\ 699391 & 349184 & 1 \end{bmatrix}$$