HOMEWORK #3 Answers (Chapter 2 First –Order Differential Equations)

In problem 1, solve the given initial-value problem.

(1) $y^{1/2}\dfrac{dy}{dx}+y^{3/2}=1,\quad y(0)=4$

Ans: 令 $\bar{y}=y^{3/2}$, $\dfrac{d\bar{y}}{dx}=\dfrac{3}{2}y^{1/2}y'$, $\therefore \dfrac{2}{3}\dfrac{d\bar{y}}{dx}+\bar{y}=1\rightarrow\dfrac{d\bar{y}}{dx}+\dfrac{3}{2}\bar{y}=\dfrac{3}{2}$, $I=e^{\int\frac{3}{2}dx}=e^{\frac{3}{2}x}$

$$e^{\frac{3}{2}x}\frac{d\bar{y}}{dx}+\frac{3}{2}e^{\frac{3}{2}x}\bar{y}=\frac{3}{2}e^{\frac{3}{2}x}\rightarrow e^{\frac{3}{2}x}\bar{y}=e^{\frac{3}{2}x}+c_1,\ \because y(0)=4\quad\therefore c_1=7\quad\therefore y^{3/2}=1+7e^{-\frac{3}{2}x}$$

In problem 2, solve the given Bernoulli equation by using an appropriate substitution

(2) $x\dfrac{dy}{dx}+y=\dfrac{1}{y^2}$

Ans: $y^2y'+\dfrac{1}{x}y^2y=\dfrac{1}{x}$, 令 $\bar{y}=y^3\rightarrow\bar{y}'=3y^2y'$, $\therefore\dfrac{1}{3}\bar{y}'+\dfrac{1}{x}\bar{y}=\dfrac{1}{x}\rightarrow\bar{y}'+\dfrac{3}{x}\bar{y}=\dfrac{3}{x}$

$$I=e^{\int\frac{3}{x}dx}=x^3,\ \therefore x^3\bar{y}'+3x^2\bar{y}=3x^2\rightarrow x^3\bar{y}=x^3+c_1\rightarrow y^3=1+\frac{c_1}{x^3}$$

(3) The differential equation $\dfrac{dy}{dx}=P(x)+Q(x)y+R(x)y^2$ is known as Ricatti's equation.

(a) A Ricatti equation can be solved by a succession of two substitutions provided we know a particular solution y_1 of the equation. Show that the substitution $y=y_1+u$ reduces Ricatti's equation to a Bernoulli equation (4) with $n=2$. The Bernoulli equation can that be reduced to a linear equation by the substitution $w=u^{-1}$.

Ans: $y=y_1+u\rightarrow\dfrac{dy}{dx}=\dfrac{dy_1}{dx}+\dfrac{du}{dx}$

$$\therefore\frac{dy}{dx}=P(x)+Q(x)y+R(x)y^2\rightarrow\frac{dy_1}{dx}+\frac{du}{dx}=P(x)+Q(x)(y_1+u)+R(x)(y_1+u)^2$$

$$\frac{dy_1}{dx}=P(x)+Q(x)y_1+R(x)y_1^2\quad\text{(Ricatti's eq.)}$$

$$\frac{du}{dx}=[Q(x)+2y_1R(x)]u+R(x)u^2\quad\text{(Bernoulli eq. } n=2\text{)}$$

Let $w=u^{-1}$, $w'=-u^{-2}u'$

$$\frac{du}{dx}=[Q(x)+2y_1R(x)]u+R(x)u^2\rightarrow(-u^2)w'-[Q(x)+2y_1R(x)]u=R(x)u^2$$

$$\therefore w'+[Q(x)+2y_1R(x)]u^{-1}=-R(x)\rightarrow w'+[Q(x)+2y_1R(x)]w=-R(x)$$

(b) Find a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx}=-\frac{4}{x^2}-\frac{1}{x}y+y^2.$$

Ans: 由 $y_1=\frac{2}{x}$ 知滿足 $\frac{dy}{dx}=-\frac{4}{x^2}-\frac{1}{x}y+y^2$ ，所以 $y_1=\frac{2}{x}$ 爲其一解。

故 $P(x)=\frac{-4}{x^2}$，$Q(x)=\frac{-1}{x}$，$R(x)=1$，$y_1=\frac{2}{x}$ 代入(a)結果可得

$$\frac{du}{dx}=\left[\frac{-1}{x}+\frac{4}{x}\right]u+u^2=\frac{3}{x}u+u^2$$

Let $w=u^{-1}$，$w'=-u^{-2}u'$

$$w'+\frac{3}{x}w=-1,\quad I=e^{\int\frac{3}{x}dx}=x^3\quad \therefore x^3w'-3x^2w=x^3\rightarrow w=\frac{-1}{4}x+\frac{c}{x^3}$$

$$u=\left[\frac{-1}{4}x+\frac{c}{x^3}\right]^{-1}$$

$$y=y_1+u=\frac{2}{x}+\left[\frac{-1}{4}x+\frac{c}{x^3}\right]^{-1}$$