HOMEWORK #4 (Chapter 2 First –Order Differential Equations)

1. Suppose $M(x,y)dx+N(x,y)dy=0$ is a nonexact differential equation. Under certain conditions integrating factors (see Problem 29 and 30) can be found in a straightforward manner. If $(M_y-N_x)/N=p(x)$, that is a function of x alone, then an integrating factor is $\mu(x)=e^{\int p(x)dx}$. If $(N_x-M_y)/M=q(y)$, that is a function of y alone, then an integrating factor is $\mu(y)=e^{\int q(y)dy}$. In Problem (a),(b) (33,34) ,solve the given differential equation by finding an appropriate integrating factor.

(a) $6xydx+(4y+9x^2)dy=0$

Ans: $N_x=18x$, $M_y=6x$, $q(y)=\dfrac{(N_x-M_y)}{M}=\dfrac{2}{y}$, $\therefore \mu(y)=e^{\int \frac{2}{y}dy}=y^2$

$$\frac{\partial\phi}{\partial x}=\mu M\to\phi=3x^2y^3+h(y),\ \frac{\partial\phi}{\partial y}=\mu N\to 9x^2y^2+h'(y)=4y^3+9x^2y^2$$

$h(x)=y^4$, $\therefore 3x^2y^3+y^4=c$

(b) $\cos xdx+(1+\dfrac{2}{y})\sin xdy=0$

Ans: $N_x=(1+\dfrac{2}{y})\cos x$, $M_y=0$, $p(x)=\dfrac{(M_y-N_x)}{N}=-\cot x$, $\therefore \mu(x)=e^{-\int\cot xdx}=\csc x$

$$\frac{\partial\phi}{\partial x}=\mu M\to\phi=\ln|\sin x|+h(y),\ \frac{\partial\phi}{\partial y}=\mu N\to h'(y)=(1+\frac{2}{y})\to h(y)=y+2\ln|y|$$

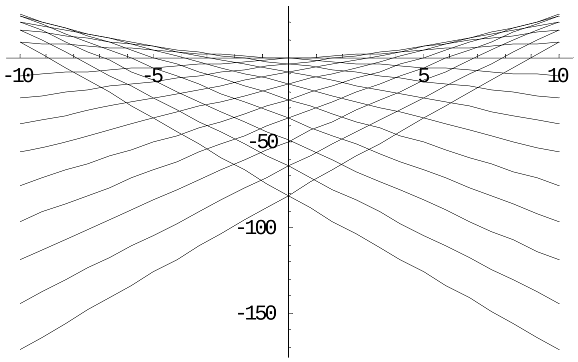
$\therefore \ln|\sin x|+y+2\ln|y|=c$

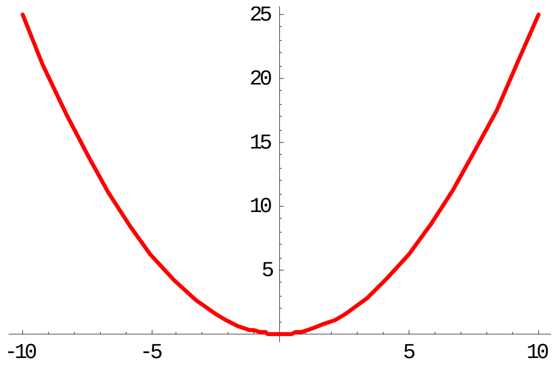
2. Solve $(y')^2-xy'+y=0$ and plot the graph of general solution and singular solution.

Ans: Let $p=y'$, $y=xp-p^2\to y'=p+xp'-2pp'\to\dfrac{dp}{dx}(x-2p)=0$

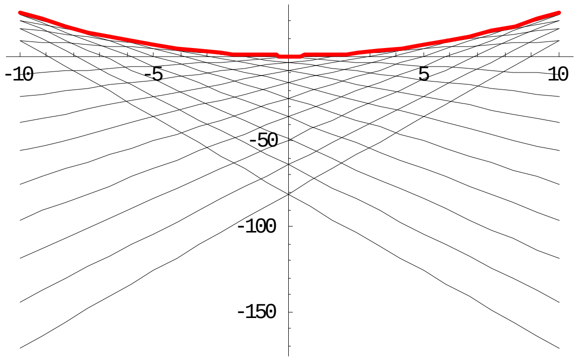
General solution 解$\begin{cases}\dfrac{dp}{dx}=0\\ y=xp-p^2\end{cases}$, 由$\dfrac{dp}{dx}=0\to p=c$, 由$y=xp-p^2\to y=cx-c^2$.

Singular solution 解$\begin{cases}x-2p=0\\ y=xp-p^2\end{cases}$, 由$x-2p=0\to p=\dfrac{x}{2}$, 由$y=xp-p^2\to y=\dfrac{x^2}{4}$.





General solution　　　　　　　　　　Singular solution



3. A Bernoulli equation of $y' + xy = xy^2$,

(a) Homeogeneous (Yes or No), why?

Ans: Yes，因爲無因變數零次項。

(b) O.D.E or P.D.E, why?

Ans: O.D.E，因爲只有一個自變數。

(c) Linear or nonlinear, why?

Ans: nonlinear，因爲有因變數的二次項(y^2)。

(d) First-order or Second-order, why?

Ans: First-order，因爲可表示爲 $F(x, y, y') = 0$ 。

(e) Separable (Yes or No), why?

Ans: Yes，因爲可表示爲 $y' = A(x)B(x)$ 。

(f) Exact (Yes or No), why?

Ans: No， $\dfrac{\partial M(x,y)}{\partial y} \neq \dfrac{\partial N(x,y)}{\partial x}$ 。

(g) Solve by Separable variable method.

Ans: $\dfrac{dy}{dx} = x(y^2 - y) \rightarrow \int \dfrac{1}{(y^2 - y)} dy = \int x dx \rightarrow -\ln|y| + \ln|y-1| = \dfrac{1}{2}x^2 + c$ 。

(h) Solve by linear O.D.E method (Convert the O.D.E to a linear equation by using the change of variable method).

Ans: $y^{-2}y' + xy^{-1} = x$ Let $u = y^{-1}$, $u' = -y^{-2}y'$

$$\therefore u' + xu = x \to \int \frac{1}{(1-u)}du = \int xdx \to -\ln|1-u| = \frac{1}{2}x^2 + c \to -\ln|1-y^{-1}| = \frac{1}{2}x^2 + c$$

(i) Solve by the Exact O.D.E method.

Ans: $xy - xy^2 + y' = 0$, $M = xy - xy^2 \to \frac{\partial M}{\partial y} = x - 2xy$, $N = 1 \to \frac{\partial N}{\partial x} = 0$, Not Exact

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \to \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x} \to \mu x(1-2y) + x(y-y^2)\frac{\partial \mu}{\partial y} = 0 + \frac{\partial \mu}{\partial x}$$

Let $\mu = \mu(y)$, $\mu(1-2y) + (y-y^2)\frac{\partial \mu}{\partial y} = 0 \to \mu = \frac{1}{y-y^2}$

$$\frac{\partial \phi}{\partial x} = \mu M = \frac{xy-xy^2}{y-y^2} = x,\ \frac{\partial \phi}{\partial y} = \mu N = \frac{1}{y} + \frac{1}{1-y} \to \phi = \ln|y| + \ln|1-y| + h(x)$$

$$\frac{\partial \phi}{\partial x} = h'(x) = x \to h(x) = \frac{x^2}{2} \quad \therefore \ln|y| + \ln|1-y| + \frac{x^2}{2} = c$$

4. The Ricatti equation $y' = \frac{1}{2x}y^2 - \frac{1}{x}y - \frac{4}{x}$ by using the solution $y_2 = y_1 + \frac{1}{z}$ with $y_1 = 4$, we obtain $y_2 = 4 + \frac{6x^3}{C - x^3}$, $C \in R$.

(a) By setting C=0, we have $y_2 = -2$, solve $y_3 = -2 + \frac{1}{z}$, please find y_3.

Ans: $y_3 = -2 + \frac{1}{z} \to y_3' = -z^{-2}z'$, $-z^{-2}z' = \frac{1}{2x}(-2+\frac{1}{z})^2 - \frac{1}{x}(-2+\frac{1}{z}) - \frac{4}{x} \to z' - \frac{3}{x}z = \frac{-1}{2x}$

$$I = x^{-3},\ \therefore x^{-3}z' - 3x^{-4}z = \frac{-1}{2}x^{-4} \to \frac{d}{dx}(x^{-3}z) = \frac{-1}{2}x^{-4} \to z = \frac{1}{6} + cx^3,$$

$$\therefore y_3 = -2 + (\frac{1}{6} + cx^3)^{-1}$$

(b) $y_3 = y_2$, Yes or No?

Ans: $y_3 \neq y_2$

(c) The Ricatti equation is linear or nonlinear?

Ans: The Ricatti equation is nonlinear.