HOMEWORK #8 (Chapter 3 Higher –Order Differential Equations)

Solve the given differential equation by undetermined coefficients.

1. $y'' + y = 2x\sin x$ (Exercises 3.4 Problem 15)

Ans: $\lambda^2 + 1 = 0$, $\lambda = \pm i$, $y_c = c_1\cos x + c_2\sin x$

Let $y_p = (ax^2 + bx)\cos x + (cx^2 + dx)\sin x$

$$y_p'' = (2a + 4cx + 2d - ax^2 + bx)\cos x + (2c - 4ax - cx^2 - dx)\sin x$$

$2a + 4cx + 2d - ax^2 + bx + ax^2 + bx = 2x$, $a = \frac{-1}{2}$, $c = 0$

$2c - 4ax - cx^2 - dx + cx^2 + dx = 0$, $b = 0$, $d = \frac{1}{2}$

$$y_p = \frac{-1}{2}x^2\cos x + \frac{1}{2}x\sin x$$

$$\therefore y = c_1\cos x + c_2\sin x - \frac{1}{2}x^2\cos x + \frac{1}{2}x\sin x$$

2. $y^{(4)} - y'' = 4x + 2xe^{-x}$ (Exercises 3.4 Problem 26)

Ans: $\lambda^4 - \lambda^2 = 0$, $\lambda = \pm 1, 0, 0$, $y_c = c_1e^x + c_2e^{-x} + c_3 + c_4x$

Let $y_p = ax^3 + bx^2 + (cx^2 + dx)e^{-x}$

$$y_p'' = 6ax + 2b + 2ce^{-x} - 2(2cx + d)e^{-x} + (cx^2 + dx)e^{-x}$$

$$y_p^{(4)} = 12ce^{-x} - 4(2cx + d)e^{-x} + (cx^2 + dx)e^{-x}$$

$-(6ax + 2b) = 4x$, $a = \frac{-2}{3}$, $b = 0$

$10ce^{-x} - 2(2cx + d)e^{-x} = 2xe^{-x}$, $c = \frac{-1}{2}$, $d = \frac{-5}{2}$

$$y_p = \frac{-2}{3}x^3 + (\frac{-1}{2}x^2 - \frac{5}{2}x)e^{-x}$$

$$y = c_1e^x + c_2e^{-x} + c_3 + c_4x - \frac{2}{3}x^3 + (\frac{-1}{2}x^2 - \frac{5}{2}x)e^{-x}$$

Solve the given initial-value problem.

3. $\dfrac{d^2x}{dt^2}+\omega^2 x = F_0 \sin\omega t$, $x(0)=0$, $x'(0)=0$ (Exercises 3.4 Problem 33)

Ans: $\lambda^2+\omega^2=0$, $\lambda=\pm\omega i$, $x_c = c_1\cos\omega t + c_2 \sin\omega t$

Let $x_p = at\cos\omega t + bt\sin\omega t$

$$x_p'' = -2a\omega\sin\omega t - a\omega^2 t\cos\omega t + 2b\omega\cos\omega t - b\omega^2 t\sin\omega t$$

$$-2a\omega - b\omega^2 t + b\omega^2 t = F_0 ,\quad a = \frac{-F_0}{2\omega}$$

$$-a\omega^2 t + 2b\omega + a\omega^2 t = 0 ,\quad b=0$$

$$x_p = \frac{-F_0}{2\omega} t\cos\omega t$$

$$x = c_1\cos\omega t + c_2\sin\omega t - \frac{F_0}{2\omega}t\cos\omega t$$

$$x' = -c_1\omega\sin\omega t + c_2\omega\cos\omega t - \frac{F_0}{2\omega}\cos\omega t + \frac{F_0}{2}t\sin\omega t$$

$$x(0)=0 \to c_1 = 0 ,\qquad x'(0)=0 \to c_2 = \frac{F_0}{2\omega^2}$$

$$\therefore x = \frac{F_0}{2\omega^2}\sin\omega t - \frac{F_0}{2\omega}t\cos\omega t$$

Solve the given differential equation by using the variation of parameters.

4. $x^2y'' + 10xy' + 8y = x^2$ (Exercises 3.6 Problem 29)

Ans: $m^2+9m+8=0$, $m=-8,-1$, $y_c = c_1x^{-8}+c_2x^{-1}$

Let $y_p = u_1x^{-8}+u_2x^{-1}$, $\quad w = 7x^{-10}$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}}{7x^{-10}} = \frac{-1}{7}x^9 ,\qquad u_1 = \int u_1'\,dx = \frac{-1}{70}x^{10}$$

$$u_2' = \frac{\begin{vmatrix} x^{-8} & 0 \\ -8x^{-9} & 1 \end{vmatrix}}{7x^{-10}} = \frac{1}{7}x^2 ,\qquad u_2 = \int u_2'\,dx = \frac{1}{21}x^3$$

$$y_p = \frac{-1}{70}x^2 + \frac{1}{21}x^2 = \frac{1}{30}x^2$$

$$y = c_1x^{-8} + c_2x^{-1} + \frac{1}{30}x^2$$

5. $x^2y'' - 3xy' + 13y = 4 + 3x$ (Exercises 3.6 Problem 31)

Ans: Let $t = \ln x$, $x = e^t$

$$\therefore x^2y'' - 3xy' + 13y = 4 + 3x \rightarrow Y''(t) - 4Y'(t) + 13Y(t) = 4 + 3e^t$$

$$\lambda^2 - 4\lambda + 13 = 0,\ \lambda = 2 \pm 3i,\ \therefore y_c = e^{2t}[c_1\cos(3t) + c_2\sin(3t)]$$

Let $y_p = u_1e^{2t}\cos(3t) + u_2e^{2t}\sin(3t)$, $w = 3e^{4t}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{2t}\sin(3t) \\ 4+3e^t & 2e^{2t}\sin(3t) + 3e^{2t}\cos(3t) \end{vmatrix}}{3e^{4t}} = \frac{-4}{3}e^{-2t}\sin(3t) - e^{-t}\sin(3t)$$

$$u_1 = \int u_1'\,dt = \frac{4}{13}e^{-2t}\cos(3t) + \frac{8}{39}e^{-2t}\sin(3t) + \frac{3}{10}e^{-t}\cos(3t) + \frac{1}{10}e^{-t}\sin(3t)$$

$$u_2' = \frac{\begin{vmatrix} e^{2t}\cos(3t) & 0 \\ 2e^{2t}\cos(3t) - 3e^{2t}\sin(3t) & 4+3e^t \end{vmatrix}}{3e^{4t}} = \frac{4}{3}e^{-2t}\cos(3t) + e^{-t}\cos(3t)$$

$$u_2 = \int u_2'\,dt = \frac{4}{13}e^{-2t}\sin(3t) - \frac{8}{39}e^{-2t}\cos(3t) + \frac{3}{10}e^{-t}\sin(3t) - \frac{1}{10}e^{-t}\cos(3t)$$

$$y_p = u_1e^{2t}\cos(3t) + u_2e^{2t}\sin(3t) = \frac{4}{13} + \frac{3}{10}e^t$$

$$\therefore y = e^{2t}[c_1\cos(3t) + c_2\sin(3t)] + \frac{4}{13} + \frac{3}{10}e^t = x^2[c_1\cos(3\ln x) + c_2\sin(3\ln x)] + \frac{4}{13} + \frac{3}{10}x$$