Final Exam

Jan. 2005

1) Solve $4x^2 y'' + 2xy' - xy = 0$ by using the method of **Frobenius** (25 scores)

a) show that zero is a regular singular point of the differential equation

b) solve the indicial equation

c) determine the recurrence relation

d) find the Frobenius solution (the **first five** terms) based on r_2, the smaller root of the indicial equation

a)

➔ $P(x) = 4x^2$, $Q(x) = 2x$, $R(x) = -x$

➔ $Q(x)/P(x) = 1/(2x)$, $R(x)/P(x) = -1/(4x)$

➔ $Q(x)/P(x) = 1/(2x)$, $R(x)/P(x) = -1/(4x)$ fail to be analytic at $x = 0$

($\because$ the denominators are zero at $x = 0$)➔ $x = 0$ is a singular point.

➔ $xQ(x)/P(x) = 1/2$, $x^2 R(x)/P(x) = -x/4$

➔ $xQ(x)/P(x) = 1/2$, $x^2 R(x)/P(x) = -x/4$ are both analytic at $x = 0$

$x = 0$ is a regular singular point of the differential equation.

b)

By *Method of Frobenius*, there exists at least one Frobenius solution

$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$

➔ $y'(x) = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$, $y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$

➔ $4\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + 2\sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$

➔ $4\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + 2\sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} - \sum_{n=1}^{\infty} c_{n-1} x^{n+r} = 0$

➔ $[4r(r-1) + 2r]\ c_0 x^r + \sum_{n=1}^{\infty} [4(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - c_{n-1}] x^{n+r} = 0$

Since $c_0 \neq 0$ as part of the Frobenius method

➔we get the indicial equation $4r(r-1) + 2r = 0$ ➔ $r = 1/2,\ \ 0$

c)

➔the recurrence relation is

$$4(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - c_{n-1} = 0$$

$$\Longrightarrow c_n = \frac{1}{2(n+r)[2(n+r)-1]} c_{n-1} \qquad \text{for } \ n \geq 1$$

d)

For $r = 0$

$\Rightarrow c_n = \dfrac{1}{2n(2n-1)} c_{n-1} \Rightarrow$

$$c_1 = \frac{1}{2\cdot 1} c_0 = \frac{1}{2!} c_0, \quad c_2 = \frac{1}{4\cdot 3} c_1 = \frac{1}{4!} c_0, \quad c_3 = \frac{1}{6\cdot 5} c_2 = \frac{1}{6!} c_0, \quad c_4 = \frac{1}{8!} c_0$$

$$\Rightarrow y_2 = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^n = c_0 (1 + \frac{1}{2!} x + \frac{1}{4!} x^2 + \frac{1}{6!} x^3 + \frac{1}{8!} x^4 + ...)$$

2) Solve $x^2 y'' + 2xy' - 6y = 0$ by using the method of **Frobenius** (25 scores)

a) point out all the singular point(s) of the differential equation

b) solve the indicial equation

c) determine the recurrence relation

d) find the general solution using the results of a) and b)

a) zero is a (regular) singular point of the differential equation

b)

$$\Rightarrow y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)\ c_n x^{n+r} - \sum_{n=0}^{\infty} 6c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - 6c_n] x^{n+r} = 0$$

$$\Rightarrow [r(r-1)\ c_0 + 2r\ c_0 - 6c_0] x^r + \sum_{n=1}^{\infty} [(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - 6c_n] x^{n+r} = 0$$

$c_0 \neq 0 \ \Rightarrow r^2 + r - 6 = 0, \ r_1 = 2, \ r_2 = -3$

c) $(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - 6c_n = (n+r+3)(n+r-2)c_n = 0$,

for $r_1 = 2, \ r_2 = -3, \ n \geq 1 \Rightarrow \ (n+r+3)(n+r-2) \neq 0$

$\Rightarrow c_n = 0, \ n \geq 1$

d) $y = cx^2 + c^* x^{-3}$

3) a) Let $L[f(t)](s) = F(s)$, show $L[t\ f(t)](s) = -F'(s)$

(hint: $L[f](s) = \int_0^{\infty} e^{-st} f(t) dt$) (5 scores)

b) Known $L\left[f^{'}\right](s) = sF(s) - f(0)$, show $L\left[f^{''}\right](s) = s^2F(s) - sf(0) - f^{'}(0)$ (5 scores)

c) what is the Dirac Delta function $\delta(t)$ (5 scores)

d) Solve $y^{''} + 2ty^{'} - 2y = \delta(t); \quad y(0) = y^{'}(0) = 0$ by applying the Laplace transform

(hint: $\lim_{s\to\infty} F(s) = 0$) (15 scores)

a) See *Theorem* 3.13 in the textbook.

b) Given $L\left[f^{'}\right](s) = sF(s) - f(0)$

➔ $L\left[f^{''}\right](s) = s\,L\left[f^{'}\right](s) - f^{'}(0)$

$$= s[sF(s) - f(0)] - f^{'}(0)$$

$$= s^2\,F(s) - sf(0) - f^{'}(0)$$

c) Let $\delta_\varepsilon(t) = \frac{1}{\varepsilon}[H(t) - H(t-\varepsilon)]$, a pulse of magnitude $1/\varepsilon$ and duration ε

➔ $\delta(t) = \lim_{\varepsilon\to 0^+} \delta_\varepsilon(t)$, a pulse of infinite magnitude over an infinitely short duration

d)

➔ $s^2Y(s) - sy(0) - y^{'}(0) - 2Y(s) - 2sY^{'}(s) - 2Y(s) = 1$

➔ $s^2Y(s) - 4Y(s) - 2sY^{'}(s) = 1, \quad Y^{'} + \left(\frac{2}{s} - \frac{s}{2}\right)Y = -\frac{1}{2s}$

To find an integrating factor

➔ $\int \left(\frac{2}{s} - \frac{s}{2}\right) ds = 2\ln(s) - \frac{1}{4}s^2, \quad ==> e^{2\ln(s) - s^2 / 4} = s^2e^{-s^2/4}$

➔ $\left(s^2e^{-s^2/4}Y\right)^{'} = -\frac{se^{-s^2/4}}{2}$ ➔ $s^2e^{-s^2/4}Y = e^{-s^2/4} + C$

➔ $Y = \frac{1}{s^2} + \frac{Ce^{s^2/4}}{s^2}$

➔ $Y = \frac{1}{s^2} \quad (\because \lim_{s\to\infty} F(s) = 0)$

➔ $y(t) = t$

4) Consider $y^{''} + 4y^{'} + 4y = f(t), \quad y(0) = 0, \quad y^{'}(0) = 0$

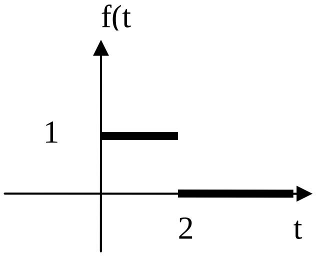
$$f(t) = \begin{cases} 1 & \quad 0 \le t < 2 \\ 0 & \textit{for} \quad t \ge 2 \end{cases}$$

a) plot the graph of $f(t)$ (5 scores)

b) what is the Heaviside function $H(t)$ (5 scores)

c) describe $f(t)$ in terms of the Heaviside function (5 scores)

d) solve the initial value problem by using the Laplace transform (15 scores) a)



b) See *Definition* 3.4 in the textbook.

c)

$f(t) = H(t) - H(t-2)$

d)

➔ $y'' + 4y' + 4y = H(t) - H(t-2)$

➔ $s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = \dfrac{1}{s} - \dfrac{e^{-2s}}{s}$

➔ $Y(s) = \dfrac{1}{s(s+2)^2}(1 - e^{-2s})$

➔ $Y(s) = \dfrac{1}{4s} - \dfrac{1}{4(s+2)} - \dfrac{1}{2(s+2)^2} - \left[\dfrac{1}{4s} - \dfrac{1}{4(s+2)} - \dfrac{1}{2(s+2)^2}\right]e^{-2s}$

➔ $y(t) = \dfrac{1}{4} - \dfrac{1}{4}e^{-2t} - \dfrac{1}{2}te^{-2t} + \left[-\dfrac{1}{4} + \dfrac{1}{4}e^{-2(t-2)} + \dfrac{1}{2}(t-2)e^{-2(t-2)}\right]H(t-2)$