1) *Section 2.3 Problems* 1. (p. 76)

$y'' + 4y = 0; \quad y_1(x) = \cos(2x)$

$y_2(x) = u(x)y_1(x) = u(x)\cos(2x)$

$y_2'(x) = u'(x)\cos(2x) - 2u(x)\sin(2x)$

$y_2'' = u''(x)\cos(2x) - 2u'(x)\sin(2x) - 2u'(x)\sin(2x) - 4u(x)\cos(2x)$

$\Longrightarrow u''(x)\cos(2x) - 4u'(x)\sin(2x) = 0$

Let $v(x) = u'(x)$

Set

$\Longrightarrow v'(x)\cos(2x) = 4v\sin(2x), \quad \frac{dv}{v} = 4\tan(2x)dx, \quad \ln|v| = 2\ln|\sec(2x)| + C$

$\Longrightarrow v = \sec^2(2x) \quad$ *Set* $\quad C = 0$

$\Longrightarrow u' = \sec^2(2x), \quad \frac{du}{dx} = \sec^2(2x), \quad u = \tan(2x)$

$\Longrightarrow y = c_1\cos(2x) + c_2\tan(2x)\cos(2x) = c_1\cos(2x) + c_2\sin(2x)$

2) *Section 2.3 Problems* 15. (p.76)

a) $yy'' + 3(y')^2 = 0$

Set $u = \frac{dy}{dx}$

➔ $y'' = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$ ➔ $yu\frac{du}{dy} + 3(u)^2 = 0$

➔ $\frac{du}{u} = -\frac{3dy}{y}$ a separable equation

➔ $\ln|u| = -3\ln|y| + C, \quad \ln|u| + 3\ln|y| = C, \quad \ln|uy^3| = C, \quad uy^3 = \pm e^C = K$

➔ $y^3\frac{dy}{dx} = K, \quad y^3 dy = Kdx$, ➔ $y^4 = 4Kx + 4C = c_1 x + c_2$

Now it's your turn to solve b) and d)

b) $yy'' + (y+1)(y')^2 = 0$

Set $u = \frac{dy}{dx}$

$$yu\frac{du}{dy} + (y+1)(u)^2 = 0, \quad \frac{du}{u} = -\frac{y+1}{y}dy, \quad u \neq 0, \quad y \neq 0$$

$$\Longrightarrow \ln|u| = -y - \ln|y| + C, \quad \ln|yu| = -y + C, \quad yu = \pm e^{-y+C} = Ke^{-y}$$

$$y\frac{dy}{dx} = Ke^{-y}, \quad ye^{y}dy = Kdx \Longrightarrow \int ye^{y}dy = \int Kdx$$

$$\Longrightarrow ye^{y} - \int e^{y} = Kx + C, \quad (y-1)e^{y} = Kx + C$$

$$\Longrightarrow (y-1)e^{y} = C_1 x + C_2$$

CHECK ! $u = 0 \Longrightarrow y = C_3$ is also a solution.

d)approach I (14. (e))

$$u = \frac{dy}{dx}$$

$$\Longrightarrow \frac{du}{dx} = y'' = 1 + (y') = 1 + u^2$$

$$\frac{du}{1+u^2} = dx, \quad \tan^{-1}(u) = x + C$$

$$u = \tan(x+C)$$

$$dy = \tan(x+C)dx$$

$$y = \ln|\sec(x+C)| + K$$

$$\Longrightarrow y = \ln|\sec(x+C_1)| + C_2$$

approach II

$$u = \frac{dy}{dx}$$

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$$

$$\Longrightarrow u\frac{du}{dy} = 1 + u^2, \quad \frac{udu}{1+u^2} = dy, \quad \ln|1+u^2| = 2y + C$$

$$\Longrightarrow 1 + u^2 = e^{2y+C}$$

$$Also \quad 1 + u^2 = \frac{du}{dx}, \Longrightarrow \quad u = \tan(x+K)$$

$$\Longrightarrow 1 + \tan^2(x+K) = \sec^2(x+K) = e^{2y+C}, \quad 2y + C = \ln|\sec^2(x+K)|$$

$$\Longrightarrow y = \ln|\sec(x+C_1)| + C_2$$

3) Find a second-order differential equation having the function as general solution (*Section 2.4 Problems* 32.) (p.82)

a) $c_1 e^{-2x} + c_2 e^{3x}$

➔ the roots of the *characteristic equation* are $\lambda_1 = -2$, $\lambda_2 = 3$

➔ the *characteristic equation* is $(\lambda+2)(\lambda-3)=0$, $\lambda^2 - \lambda - 6 = 0$

➔the differential equation is $y'' - y' - 6y = 0$

Now it is your turn to solve b) and c)

b) $c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x)$

➔ $c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x) = e^{-3x}\left[c_1\cos(2x) + c_2\sin(2x)\right]$

➔ the roots of the *characteristic equation* are $\lambda_1 = -3+2i$, $\lambda_2 = -3-2i$

➔ the *characteristic equation* is

$(\lambda+3-2i)(\lambda+3+2i)=0$, $\lambda^2 + 6\lambda + 13 = 0$

➔the differential equation is $y'' + 6y' + 13y = 0$

c) $c_1 e^{-4x} + c_2 x e^{-4x}$

➔ the roots of the *characteristic equation* are $\lambda_1 = -4$, $\lambda_2 = -4$

➔ the *characteristic equation* is $(\lambda+4)(\lambda+4)=0$, $\lambda^2 + 8\lambda + 16 = 0$

➔the differential equation is $y'' + 8y' + 16y = 0$

4) *Section 2.5 Problems* 1. (p.85)

$x^2 y'' + 2xy' - 6y = 0$

$Y'' + (A-1)Y' + BY = 0$

➔ $Y'' + Y' - 6Y = 0$

$Y(t) = c_1 e^{2t} + c_2 e^{-3t}$

➔ $y(x) = c_1 x^2 + c_2 x^{-3}$