1) *Sections 2.2 Problems*   5. (p. 73)

$$y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0, \quad y(1) = 2, \quad y'(1) = 4$$

Approach I

$$x = e^t, \quad t = \ln(x), \quad x > 0$$

$$Y(t) = y(e^t)$$

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = Y'(t)\frac{1}{x}$$

$$Y'(t) = xy'(x)$$

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right)$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{d}{dx}Y'(t)$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'}{dt}\frac{dt}{dx}$$

$$= \frac{1}{x^2}\left(Y''(t) - Y'(t)\right)$$

==> $x^2 y''(x) = Y''(t) - Y'(t)$

$$Y'' - 8Y' + 16Y = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

➔ $\lambda = 4, \quad 4$

$$Y_1 = e^{4t}, \quad Y_2 = te^{4t}$$

$$y_1 = e^{4\ln(x)} = x^4, \quad y = x^4 \ln(x)$$

➔ $W(x) = \begin{vmatrix} x^4 & x^4 \ln(x) \\ 4x^3 & 4x^3 \ln(x) + x^3 \end{vmatrix} = 4x^7 \ln(x) + x^7 - 4x^7 \ln(x) = x^7 \neq 0 \qquad (x > 0)$

$$y = C_1 x^4 + C_2 x^4 \ln(x)$$

➔ $y(1) = C_1 + C_2 \ln(1) = 2, \quad C_1 = 2$

$$y'(1) = 4C_1 + C_2 = 4, \quad C_2 = -4$$

$$y = 2x^4 - 4x^4 \ln(x)$$

Approach II (See 26. page 86)

$y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$\Longrightarrow r(r-1)x^{r-2} - 7rx^{r-2} + 16x^{r-2} = 0$

$r(r-1) - 7r + 16 = 0$

$r^2 - 8r + 16 = 0$

$r = 4, \quad 4$

$y_1 = x^4$

$y_2 = u(x)y_1, \quad \Longrightarrow y_2 = x^4 \ln(x), \quad (x > 0)$

2) *Sections 2.3 Problems* 8. (p. 76)

$$y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0, \quad y_1 = x$$

$y_2 = u(x)y_1$

$y'_2 = u'x + u$

$y''_2 = u''x + 2u'$

$$u''x + 2u' - \frac{2x}{1+x^2}\left(u'x + u\right) + \frac{2}{1+x^2}ux = 0$$

$$u''x + 2u' - \frac{2x^2}{1+x^2}u' = 0$$

$$u''x + \frac{2}{1+x^2}u' = 0$$

$v = u'$

$$v' = \frac{-2}{x(1+x^2)}v$$

$$\frac{dv}{v} = \left(\frac{a}{x} + \frac{bx+c}{1+x^2}\right)dx, \quad \Longrightarrow a = -2, \quad b = 2, \quad c = 0$$

$$\frac{dv}{v} = \left(\frac{-2}{x} + \frac{2x}{1+x^2}\right)dx$$

$$\ln|v| = -2\ln|x| + \ln(1+x^2) = \ln\left(\frac{1+x^2}{x^2}\right)$$

➔ $v = \dfrac{1+x^2}{x^2} = u', \quad du = (1 + \dfrac{1}{x^2})dx$

$$u = x - \frac{1}{x}, \quad y_2 = ux = x^2 - 1$$

or $v = -\dfrac{1+x^2}{x^2} = u', \quad y_2 = 1 - x^2$

3) *Sections 2.5 Problems* 17. (p. 86)

$$x^2 y'' + 5xy' + 20y = 0, \quad y(-1) = 3, \quad y'(-1) = 2 \quad \text{for} \quad x < 0$$

Approach I

Set $|x| = e^t$, $x = -e^t$, $t = \ln|x|$

$$Y(t) = y(-e^t)$$

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = Y'(t)\frac{1}{x}, \quad Y'(t) = xy'(x)$$

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}\left[\frac{1}{x}Y'(t)\right]$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{d}{dx}Y'(t)$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'}{dt}\frac{dt}{dx}$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}Y''\frac{1}{x}$$

$$= \frac{1}{x^2}\left[Y''(t) - Y'(t)\right]$$

$$x^2 y'' = Y''(t) - Y'(t)$$

$$Y''(t) - Y'(t) + 5Y'(t) + 20Y = 0$$

$$Y'' + 4Y' + 20Y = 0$$

$$\lambda^2 + 4\lambda + 20 = 0$$

$$\lambda = -2 \pm 4i$$

$$Y(t) = e^{-2t}\left[C_1 \cos(4t) + C_2 \sin(4t)\right]$$

$$y(x) = x^{-2}\left[C_1 \cos(4\ln|x|) + C_2 \sin(4\ln|x|)\right]$$

$$y(-1) = C_1 \cos(0) + C_2 \sin(0) = C_1 = 3$$

$$y'(x) = -2x^{-3}\left[C_1 \cos(4\ln|x|) + C_2 \sin(4\ln|x|)\right] +$$

$$x^{-2}\left[-C_1 \sin(4\ln|x|)\frac{4}{x} + C_2 \cos(4\ln|x|)\frac{4}{x}\right]$$

$$y'(-1) = 2\left[3\cos(0) + C_2 \sin(0)\right] + \left[-3\sin(0)\frac{4}{-1} + C_2 \cos(0)\frac{4}{-1}\right]$$

$$= 6 - 4C_2 = 2 \qquad ==> C_2 = -1$$

➔

$$y = x^{-2}\left[3\cos(4\ln|x|) - \sin(4\ln|x|)\right]$$

Approach II (See 26. page 86)

$$y = x^r$$
$$y' = rx^{r-1}$$
$$y'' = r(r-1)x^{r-2}$$
$$\Longrightarrow x^2 r(r-1)x^{r-2} + 5xrx^{r-1} + 20x^r = 0$$
$$r(r-1) + 5r + 20 = 0$$
$$r^2 + 4r + 20 = 0$$
$$r = -2 \pm 4i$$
$$x^{-2\pm 4i} = x^{-2}x^{\pm 4i} = x^{-2}e^{\pm 4i\ln|x|} = x^{-2}\left[\cos(4\ln|x| \pm \sin(4\ln|x|)\right]$$
$$\Longrightarrow y(x) = x^{-2}\left[C_1\cos(4\ln|x|) + C_2\sin(4\ln|x|)\right]$$
$$\Longrightarrow C_1 = 3, C_1 = -1$$

4) *Section 2.6 Problems* 62. (p.98)

$$(x^2+1)y'' - 2xy' + 2y = 0, \quad y_1 = x, \quad y_2 = x^2 - 1$$

solve $(x^2+1)y'' - 2xy' + 2y = 6(x^2+1)^2$

➔ $y'' - \dfrac{2x}{x^2+1}y' + \dfrac{2}{x^2+1}y = 6(x^2+1)$

Compared to the general form $y'' + p(x)y' + q(x)y = f(x)$,

$p(x)$, $q(x)$ are not constant, so the method of undetermined coefficients is not applicable here.

$$y_p = u(x)y_1 + v(x)y_2$$
$$W(x) = \begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 + 1 = x^2 + 1$$
$$u' = -\frac{y_2 f(x)}{W} = -6(x^2-1)$$

➔ $v' = \dfrac{y_1 f(x)}{W} = 6x$

$$u = -2x^3 + 6x$$
$$v = 3x^2$$
$$y_p = ux + v(x^2-1) = x^4 + 3x^2$$
$$\varphi = C_1 y_1 + C_2 y_2 + y_p$$

5)*Additional Problems* 9. (p.112)

$$yy'' - 2(y')^2 = 0$$

$$u = y'$$

$$y'' = \frac{du}{dx} = u\frac{du}{dy}$$

$$\Longrightarrow yu\frac{du}{dy} - 2u^2 = 0$$

$$\frac{du}{u} = \frac{2dy}{y}$$

$$\ln|u| = 2\ln|y| + C$$

$$|u| = e^C y^2, \quad u = \pm e^C y^2 = Ky^2$$

$$y' = Ky^2$$

$$\frac{dy}{y^2} = Kdx, \quad -\frac{1}{y} = Kx + c$$

$$\Longrightarrow y = \frac{1}{c_1 x + c_2}$$