1) *Sections 4.1 Problems* 17. (p. 168)

$$y' + xy = 1 - x + x^2$$

All coefficient functions are ***polynomial functions***➔ analytic everywhere

The ***Maclaurin expansion*** (*about 0*) of the ***general solution*** is

➔ $$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0)\ x^n$$
$$= y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}x^4 + \ldots$$

Let the initial condition $y(0) = A$

➔ $y'(0) + 0 \cdot y(0) = 1$, ==> $y'(0) = 1$

➔ $y'' + xy' + y = -1 + 2x$ ➔ $y''(0) + 0 \cdot y'(0) + y(0) = -1$, ==> $y''(0) = -1 - A$

➔ $y''' + xy'' + 2y' = 2$ ➔ $y'''(0) + 0 \cdot y''(0) + 2y'(0) = 2$, ==> $y'''(0) = 0$

➔ $y^{(4)} + xy''' + 3y'' = 0$ ➔ $y^{(4)}(0) + 0 \cdot y'''(0) + 3y''(0) = 0$, ==> $y^{(4)}(0) = 3 + 3A$

➔ $y^{(5)} + xy^{(4)} + 4y''' = 0$ ➔ $y^{(5)}(0) + 0 \cdot y^{(4)}(0) + 4y'''(0) = 0$ ==> $y^{(5)}(0) = 0$

➔ $y^{(6)} + xy^{(5)} + 5y^{(4)} = 0$

➔ $y^{(6)}(0) + 0 \cdot y^{(5)}(0) + 5y^{(4)}(0) = 0$ ==> $y^{(6)}(0) = -15 - 15A$

The ***Maclaurin series*** solution (only the first five terms shown) of the differential equation is

$$y(x) = A + x - \frac{1}{2}(1+A)x^2 + \frac{1}{8}(1+A)x^4 - \frac{1}{48}(1+A)x^6 + \ldots$$
$$= x - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots$$
$$+ A\left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right]$$
$$= x - 1 + \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right]$$
$$+ A\left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right]$$
$$= x - 1 + (A-1)\left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right]$$
$$\left(= x - 1 + (A-1)\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!2^n} x^{2n} = x - 1 + (A-1)e^{-x^2/2} \right)$$

2) *Sections 4.1 Problems* 31. (p. 169)

$$y'' + xy = 0; y(0) = a, \quad y'(0) = b$$

All coefficient functions are ***polynomial functions***➔ analytic everywhere

The ***Maclaurin series*** (*about 0*) solution of Airy's equation

➔
$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0)\ x^n$$
$$= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + \frac{1}{24} y^{(4)} x^4 + \ldots$$

The initial conditions give us $y(0) = a, \quad y'(0) = b$

➔ $y''(0) + 0 \cdot y(0) = 0, \quad ==> y''(0) = 0$

➔ $y''' + xy' + y = 0$ ➔ $y'''(0) + 0 \cdot y'(0) + y(0) = 0, \quad ==> y'''(0) = -a$

➔ $y^{(4)} + xy'' + 2y' = 0$ ➔ $y^{(4)}(0) + 0 \cdot y''(0) + 2y'(0) = 0 ==> \quad y^{(4)}(0) = -2b$

➔ $y^{(5)} + xy''' + 3y'' = 0$ ➔ $y^{(5)}(0) + 0 \cdot y'''(0) + 3y''(0) = 0 ==> \quad y^{(5)}(0) = 0$

➔ $y^{(6)} + xy^{(4)} + 4y''' = 0$ ➔ $y^{(6)}(0) + 0 \cdot y^{(4)}(0) + 4y'''(0) = 0 ==> \quad y^{(6)}(0) = 4a$

The ***Maclaurin series*** solution (only the first five terms shown) of the differential equation is

$$y(x) = a + bx - \frac{1}{6} ax^3 - \frac{1}{12} bx^4 + \frac{1}{180} ax^6 + \ldots$$

3) *Section 4.2 Problems* 13. (p. 174)

NOTE: point out which term the particular solution is

$$y'' + y' - y = 1 - x^2 + x^4$$

All coefficient functions are ***polynomial functions***➔ analytic everywhere

The ***Maclaurin series*** (*about 0*) of the general solution

➔ $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ (***Note: the series for*** y' ***begins at*** $n = 1$***, and that for*** y'' ***begins at*** $n = 2$***, see P.169***)

➔ $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 1 - x^2 + x^4$

Shit indices in the first and second term summation so that the power of x occurring in each is x^n

➔ $\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = 1 - x^2 + x^4$

➔ $\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_n\right]x^n = 1 - x^2 + x^4$

➔ $\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_n\right]x^n = 1 - x^2 + x^4$

For this to hold for all x in some interval abourt 0, the coefficient of x^n on the left must match the coefficient of x^n on the right.

➔

$2a_2 + a_1 - a_0 = 1, \quad 6a_3 + 2a_2 - a_1 = 0, \quad 12a_4 + 3a_3 - a_2 = -1, \quad 20a_5 + 4a_4 - a_3 = 0$

$30a_6 + 5a_5 - a_4 = 1$, Note that a_0, a_1 are arbitrary. And

$$a_n = \frac{-(n-1)a_{n-1} + a_{n-2}}{n(n-1)}, \quad for\ n \geq 7$$

The general solution of the equation is

$$y(x) = a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + ...\right] + a_1\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{24}x^5 - ...\right] +$$
$$\left[\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{5}{144}x^6 - \frac{13}{2520}x^7 + ...\right]$$

with $y(0) = a_0$, $y^{'}(0) = a_1$. Also the third term above is the particular solution of the equation.

4) *Section 4.3 Problems* 1. (p. 181)

$x^2(x-3)^2 y^{''} + 4x(x^2 - x - 6)y^{'} + (x^2 - x - 2)y = 0$

➔ $P(x) = x^2(x-3)^2, \quad Q(x) = 4x(x^2 - x - 6), \quad R(x) = (x^2 - x - 2)$

$P(x) = x^2(x-3)^2 = 0 \quad ==> x = 0, \quad x = 3$ ➔ $x = 0, \quad x = 3$ are singular points (see p.174, *Definition 4.2*)

a)For $x = 0$

$xQ(x)/P(x) = 4(x+2)/(x-3), \quad x^2R(x)/P(x) = (x^2 - x - 2)/(x-3)^2$

➔ $xQ(x)/P(x) = 4(x+2)/(x-3), \quad x^2R(x)/P(x) = (x^2 - x - 2)/(x-3)^2$ are both analytic ➔ $x = 0$ is a regular point. (see *Definition 4.3*)

b) For $x = 3$

$(x-3)Q(x)/P(x) = 4(x+2)/x, \quad (x-3)^2R(x)/P(x) = (x^2 - x - 2)/x^2$

➔ $(x-3)Q(x)/P(x) = 4(x+2)/x, \quad (x-3)^2R(x)/P(x) = (x^2 - x - 2)/x^2$ are both analytic ➔ $x = 3$ is a regular point. (see *Definition 4.3*)

5) *Section 4.3 Problems* 7. (p.181)

$4x^2 y'' + 2xy' - xy = 0$

a)

➔ $P(x) = 4x^2$, $Q(x) = 2x$, $R(x) = -xy$

➔ $Q(x)/P(x) = 1/(2x)$, $R(x)/P(x) = -1/(4x)$

➔ $Q(x)/P(x) = 1/(2x)$, $R(x)/P(x) = -1/(4x)$ fail to be analytic at $x = 0$

($\because$ the denominators are zero at $x = 0$)➔ $x = 0$ is a singular point.

➔ $xQ(x)/P(x) = 1/2$, $x^2 R(x)/P(x) = -x/4$

➔ $xQ(x)/P(x) = 1/2$, $x^2 R(x)/P(x) = -x/4$ are both analytic at $x = 0$

By *Definition 4.3*, $x = 0$ is a regular singular point of the differential equation.

b)

By *Method of Frobenius*, there exists at least one Frobenius solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

➔ $y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$, $y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$

➔ $4\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + 2\sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$

➔ $4\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + 2\sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} - \sum_{n=1}^{\infty} c_{n-1} x^{n+r} = 0$

➔ $[4r(r-1) + 2r]\ c_0 x^r + \sum_{n=1}^{\infty} [4(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - c_{n-1}] x^{n+r} = 0$

Since $c_0 \neq 0$ as part of the Frobenius method (see *Theorem 4.3*)

➔we get the indicial equation $4r(r-1) + 2r = 0$ ➔ $r = 1/2, \quad 0$

c)

➔the recurrence relation is

$$4(n+r)(n+r-1)\ c_n + 2(n+r)\ c_n - c_{n-1} = 0$$

$$\Longrightarrow c_n = \frac{1}{2(n+r)[2(n+r)-1]} c_{n-1} \qquad \text{for} \quad n \geq 1$$

d)

For $r = 1/2$

$$\Rightarrow c_n = \frac{1}{2n(2n+1)} c_{n-1} \Rightarrow$$

$$c_1 = \frac{1}{2\cdot 3} c_0 = \frac{1}{3!} c_0, \quad c_2 = \frac{1}{4\cdot 5} c_1 = \frac{1}{5!} c_0, \quad c_3 = \frac{1}{6\cdot 7} c_2 = \frac{1}{7!} c_0, \quad c_4 = \frac{1}{9!} c_0$$

$\Rightarrow$

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+1/2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{n+1/2} = c_0 (x^{1/2} + \frac{1}{3!} x^{3/2} + \frac{1}{5!} x^{5/2} + \frac{1}{7!} x^{7/2} + \frac{1}{9!} x^{9/2} + \ldots)$$

For $r = 0$

$$\Rightarrow c_n = \frac{1}{2n(2n-1)} c_{n-1} \Rightarrow$$

$$c_1 = \frac{1}{2\cdot 1} c_0 = \frac{1}{2!} c_0, \quad c_2 = \frac{1}{4\cdot 3} c_1 = \frac{1}{4!} c_0, \quad c_3 = \frac{1}{6\cdot 5} c_2 = \frac{1}{6!} c_0, \quad c_4 = \frac{1}{8!} c_0$$

$$\Rightarrow y_2 = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^n = c_0 (1 + \frac{1}{2!} x + \frac{1}{4!} x^2 + \frac{1}{6!} x^3 + \frac{1}{8!} x^4 + \ldots)$$