The second-order differential equation $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ is called Bessel's equation of order ν, for $\nu \geq 0$. Solution of Bessel's equation are called Bessel functions. (hint: Example 4.12)

1) Find one Frobenius solution of Bessel's equation of order $\nu = 1$

2) Show the Frobenius series solution is convergent by using the ratio test (hint: Theorem 4.6)

1)First of all, **zero** is a ***regular singular point*** of Bessel's equation, so attempt a Frobenius solution $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$

➔ $y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$

Upon substituting this series into Bessel's equation

➔ $\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} + (x^2 - 1)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$

➔ $\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$

Shift indices in the third summation as

Set $n + r + 2 = m + r$ ➔ $\sum_{m=2}^{\infty} c_{m-2} x^{m+r}$ ➔ $\sum_{n=2}^{\infty} c_{n-2} x^{n+r}$

Then combine terms to write

➔ $\sum_{n=0}^{\infty} (n+r)(n+r-1)\ c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\ c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$

➔ $$\begin{aligned} &[r(r-1) + r - 1]\ c_0 x^r + [r(r+1) + (r+1) - 1]\ c_1 x^{r+1} + \\ &\sum_{n=2}^{\infty} \left\{ [(n+r)(n+r-1) + (n+r) - 1]\ c_n + c_{n-2} \right\} x^{n+r} = 0 \end{aligned} \qquad (*)$$

Setting the coefficient of x^r equal to zero (remember $c_0 \neq 0$ as part of the Frobenius method), we obtain ***the indicial equation***

$r(r-1) + (r-1) = 0$ ➔ $r = 1, \quad -1$

Note: usually we have $r_1 \geq r_2$ in the Frobenius method.

On the other hand, with $r_1 = 1$, ➔ $[r(r+1)+(r+1)-1]c_1 x^{r+1} = 3c_1 x^{r+1}$

For the equation (*) to hold

➔ $c_1 = 0$

$\left[(n+r)(n+r-1)+(n+r)-1\right] c_n + c_{n-2} = 0$ for $n = 2,3,....$

➔ $c_n = -\frac{1}{n(n+2)} c_{n-2}$ (with $r_1 = 1$)

Since $c_1 = 0$, this equation yields $c_3 = c_5 = ... = c_{odd} = 0$.

For the even-indexed coefficients, write

$$
\begin{aligned}
c_{2n} &= -\frac{1}{2n(2n+2)} c_{2n-2} = -\frac{1}{2^2 n(n+1)} c_{2n-2} \\
&= -\frac{1}{2^2 n(n+1)} \frac{-1}{2(n-1)[2(n-1)+2]} c_{2n-4} \\
&= \frac{1}{2^4 n(n-1)(n+1)(n)} c_{2n-4} \\
&= ... = \frac{(-1)^n}{2^{2n} n(n-1)...(2)(1)(n+1)(n)...(2)} c_0 \\
&= \frac{(-1)^n}{2^{2n} n!(2)(3)...(n+1)} c_0 \\
&= \frac{(-1)^n}{2^{2n} n!(n+1)!} c_0
\end{aligned}
$$

➔One Frobenius solution of Bessel's equation of order $\nu = 1$ is therefore

$y_1(x) = J_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(n+1)!} x^{2n+1}$ ➔ ***Bessel's functions of the first kind of order*** $\nu = 1$**.**

Note that, we here just find one solution of Bessel's equation. You are encouraged to try for the sencond solution for excersie. ➔ *Bessel's functions of the second kind of order* $\nu = 1$.

2)By thorem 4.6 ratio test,

$$b_n = \frac{(-1)^n}{2^{2n} n!(n+1)!} x^{2n+1}$$

$$b_{n+1} = \frac{(-1)^{n+1}}{2^{2(n+1)} (n+1)!(n+2)!} x^{2(n+1)+1}$$

$$\Rightarrow \left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}}{2^{2(n+1)}(n+1)!(n+2)!} x^{2(n+1)+1}}{\dfrac{(-1)^n}{2^{2n} n!(n+1)!} x^{2n+1}}\right| = \left|\frac{-1}{2^2 (n+1)(n+2)} x^2\right|$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{-1}{2^2 (n+1)(n+2)} x^2\right| = 0$$

$\because \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L = 0 < 1$ ➔ the Frobenius series solution is convergent for all x.

Note that, you should be familiar with the ratio test.