Mid-term Exam I

Nov. 2004

1) Verify the given function is a solution of the differential equation (10 scores)

a) $y' = -\dfrac{2y+e^x}{2x}$ for $x>0$; $\varphi(x) = \dfrac{C-e^x}{2x}$

$$\varphi'(x) = -\frac{C-e^x}{2x^2} + \frac{-e^x}{2x} \Rightarrow 2x\varphi' = -\frac{C-e^x}{x} - e^x = -2\varphi - e^x$$

b) $y' = y^2 e^{-x}$, $\varphi(x) = \dfrac{1}{e^{-x}-k}$

$$\varphi'(x) = -(e^{-x}-k)^{-2}(-e^{-x}) = \frac{e^{-x}}{(e^{-x}-k)^2}, \quad \varphi(x)^2 = \frac{1}{(e^{-x}-k)^2}$$

$\Rightarrow \varphi' = \varphi^2 e^{-x}$

2) Verify by implicit differentiation that the given equation implicitly defines a solution of the differential equation (5 scores)

$y^2 + xy - 2x^2 - 3x - 2y = C$; $y - 4x - 3 + (x+2y-2)y' = 0$

$$d(y^2 + xy - 2x^2 - 3x - 2y)/dx = d(C)/dx$$
$$\Longrightarrow 2yy' + y + xy' - 4x - 3 - 2y' = 0$$
$$\Longrightarrow y - 4x - 3 + xy' + 2yy' - 2y' = 0$$
$$\Longrightarrow y - 4x - 3 + (x+2y-2)y' = 0$$

3) Consider $y' = \dfrac{y}{x} + 1$ for all $x > 0$ (20 scores)

a) get the particular solution corresponding to the initial solution y(1)=0 (15 scores)

b) draw a direction field of the differential equation and the integral curve through (1, 0) *hint*: $\ln(2) \cong 0.69$ (10 scores)

a) $y' = \dfrac{y}{x} + 1 \Longrightarrow y' - \dfrac{y}{x} = 1$

$$\Rightarrow \mu(x) = e^{\int -\frac{1}{x}dx} = e^{\ln(1/x)} = \frac{1}{x}$$

$$\Rightarrow \frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x}, \quad \left(\frac{y}{x}\right)' = \frac{1}{x}, \quad \frac{y}{x} = \int \frac{1}{x}dx = \ln(x) + C$$

$\Rightarrow y(x) = x\ln(x) + Cx$, $y(1) = \ln(1) + C = 0$, $C = 0$

$\Rightarrow y(x) = x\ln(x)$

b)omitted (see p. Section 1.1.5 of the textbook)

4) Given a family F of curves $x^2 - Ky^2 = 1$ (25 scores)

a) describe the definition of orthogonal trajectories of a given family of curves (5 scores)

b) find the family of orthogonal trajectories of the given family F of curves (15 scores)

c) plot the orthogonal families together on the x-y plane (5 scores)

a) Two families of curves, or trajectories, are orthogonal if each curve of the first family is orthogonal to each curve of the second family wherever an intersection occurs. Namely, their slopes are negative reciprocals. (p. 55~56 of the textbook)

b) $x^2 - Ky^2 = 1$ ➔ $2x - 2Kyy' = 0$, $y' = \frac{x}{Ky} = \frac{xy}{x^2-1}$ the differential equation of the family F.

➔ The family of orthogonal trajectories of the given family F has the differential equation $y' = -\frac{1}{\frac{xy}{x^2-1}} = -\frac{x^2-1}{xy}$

$$ydy = -\frac{x^2-1}{x}dx = \frac{dx}{x} - xdx$$

➔

$$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C$$

c)omitted

5) Solve $y' + y/x = \cos(x)$, $x > 0$ (15 scores)

➔ $\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$, ➔ $xy' + y = x\cos(x)$

➔ $(xy)' = x\cos(x)$

$$xy = \int x\cos(x)dx$$

$$= x\sin(x) - \int \sin(x)dx$$

$$= x\sin(x) + \cos(x)$$

➔ $y = \sin(x) + \frac{\cos(x)}{x}$

6) Solve $x - xy^2 - y' = 0$ (15 scores)

$y' = -xy^2 + x$ it is a Riccati equation with a solution $S(x) = 1$

I)Approach I

➔ $y = S(x) + \frac{1}{z} = 1 + \frac{1}{z}$

➔ $$-\frac{1}{z^2} z' = -x\left(1 + \frac{2}{z} + \frac{1}{z^2}\right) + x$$
$$= -\frac{2}{z} x - \frac{x}{z^2}$$

➔ $z' - 2xz - x = 0$, $z' = x(2z+1)$, $\frac{dz}{2z+1} = xdx$

➔ $\int \frac{dz}{2z+1} = \int xdx$, $\frac{1}{2}\ln|2z+1| = \frac{1}{2}x^2 + C$

➔ $|2z+1| = e^{x^2+2C}$, $2z+1 = \pm e^{x^2+2C} = ke^{x^2}$, $z = \frac{ke^{x^2}-1}{2}$

➔ $y = 1 + \frac{1}{z} = 1 + \frac{2}{ke^{x^2}-1} = \frac{ke^{x^2}+1}{ke^{x^2}-1}$ ------(1)

II)Approach II

$x - xy^2 - y' = 0$ ➔ $x(1-y^2) = y'$, $xdx = \frac{dy}{1-y^2} = \frac{1}{2}\left(\frac{dy}{1+y} + \frac{dy}{1-y}\right)$, $y \neq \pm 1$

➔ $\int 2xdx = \int \left(\frac{dy}{1+y} - \frac{dy}{y-1}\right)$, $x^2 + K = \ln|1+y| - \ln|y-1| = \ln\left|\frac{1+y}{1-y}\right|$

➔ $\left|\frac{1+y}{1-y}\right| = e^{x^2+K}$, $\frac{1+y}{1-y} = \pm e^{x^2+K} = Ce^{x^2}$

➔ $1 + y = Ce^{x^2} - Cye^{x^2}$, $y = \frac{Ce^{x^2}-1}{Ce^{x^2}+1}$ ------(2)

$y = \pm 1$ are solutions, too. $y = -1$ can be obtained from (2) with C=0. BUT, $y = 1$ CAN NOT be obtained from (2).

Note: (1) and (2) are equivalent. If we have $k = -C$.

7) Solve $y' = \frac{y}{x-y}$ (10 scores)

➔ $y' = \frac{\frac{y}{x}}{1-\frac{y}{x}}$, $y = ux$, $y' = u'x + u$

➔ $u'x + u = \frac{u}{1-u}$, $\frac{1-u}{u^2}du = \left(\frac{1}{u^2} - \frac{1}{u}\right)du = \frac{dx}{x}$ ➔ $u \neq$, $x \neq 0$, $y \neq 0$

$$\Rightarrow -\frac{1}{u} - \ln|u| = \ln|x| + C, \quad \ln|ux| = -(\frac{1}{u} + C)$$

$$\Rightarrow |ux| = e^{-\frac{1}{u} - C}, \quad ux = \pm e^{-C} e^{-\frac{1}{u}}$$

$$\Rightarrow ue^{\frac{1}{u}} = \frac{K}{x}, \quad \frac{y}{x} e^{\frac{x}{y}} = \frac{K}{x}$$

$$\Rightarrow ye^{\frac{x}{y}} = K$$

Also, $y = 0$ is a solution, and can not be obtained from the general solution $ye^{\frac{x}{y}} = K$.