Mid-term Exam II

Dec. 2004

1) Consider $y' = \frac{y}{x} + 1$ for all $x > 0$

a) get the particular solution corresponding to the initial solution y(1)=0 (5 scores)

b) draw the integral curve through (1, 0) *hint*: $\ln(2) \cong 0.69$ (5 scores)

a) $y' = \frac{y}{x} + 1 \implies y' - \frac{y}{x} = 1$

➔ $\mu(x) = e^{\int -\frac{1}{x}dx} = e^{\ln(1/x)} = \frac{1}{x}$

➔ $\frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x}$, $\left(\frac{y}{x}\right)' = \frac{1}{x}$, $\frac{y}{x} = \int \frac{1}{x}dx = \ln(x) + C$

➔ $y(x) = x\ln(x) + Cx$, $y(1) = \ln(1) + C = 0$, $C = 0$

➔ $y(x) = x\ln(x)$

2) Consider $y' = \frac{y}{x - y}$

a) find the general solution (5 scores)

b) Verify your answer is the general solution (3 scores)

c) Does the general solution contain all the solutions ? If not, point out the singular solution. (2 scores)

a)

➔ $y' = \frac{\frac{y}{x}}{1 - \frac{y}{x}}$, $y = ux$, $y' = u'x + u$

➔ $u'x + u = \frac{u}{1-u}$, $\frac{1-u}{u^2}du = \left(\frac{1}{u^2} - \frac{1}{u}\right)du = \frac{dx}{x}$ ➔ $u \neq$, $x \neq 0$, $y \neq 0$

➔ $-\frac{1}{u} - \ln|u| = \ln|x| + C$, $\ln|ux| = -(\frac{1}{u} + C)$

➔ $|ux| = e^{-\frac{1}{u} - C}$, $ux = \pm e^{-C}e^{-\frac{1}{u}}$

➔ $ue^{\frac{1}{u}} = \frac{K}{x}$, $\frac{y}{x}e^{\frac{x}{y}} = \frac{K}{x}$

➔ $ye^{\frac{x}{y}} = K$

b) $ye^{\frac{x}{y}}=K$ ➔ $y'e^{\frac{x}{y}}+ye^{\frac{x}{y}}\frac{1}{y}-ye^{\frac{x}{y}}\frac{x}{y^2}y'=0,\quad y'(\frac{x}{y}-1)=1,\quad y'=\frac{y}{x-y}$

c) $y=0$ is a singular solution which can not be obtained from the general solution

$ye^{\frac{x}{y}}=K$.

3) Find a differential equation having the given function as general solution

a) $c_1e^{-2x}+c_2e^{3x}$ (2 scores)

➔ the roots of the *characteristic equation* are $\lambda_1=-2,\quad \lambda_2=3$

➔ the *characteristic equation* is $(\lambda+2)(\lambda-3)=0,\quad \lambda^2-\lambda-6=0$

➔the differential equation is $y''-y'-6y=0$

b) $e^{-x}\left[c_1\cos(\sqrt{5}x)+c_2\sin(\sqrt{5}x)\right]$ (2 scores)

➔ the roots of the *characteristic equation* are $\lambda_1=-1+\sqrt{5}i,\quad \lambda_2=-1-\sqrt{5}i$

➔ the *characteristic equation* is $(\lambda+1-\sqrt{5}i)(\lambda+1+\sqrt{5}i)=0,\quad \lambda^2+2\lambda+6=0$

➔the differential equation is $y''+2y'+6y=0$

c) $e^{3x}(c_1+c_2x)$ (2 scores)

➔ the roots of the *characteristic equation* are $\lambda_1=3,\quad \lambda_2=3$

➔ the *characteristic equation* is $(\lambda-3)(\lambda-3)=0,\quad \lambda^2-6\lambda+9=0$

➔the differential equation is $y''-6y'+9y=0$

d) $c_1x^3+c_2x^3\ln(x)$ (2 scores)

➔ $x=e^t,\quad t=\ln(x),\quad ==> c_1e^{3t}+c_2te^{3t}$

➔ the roots of the *characteristic equation* are $\lambda_1=3,\quad \lambda_2=3$

➔ the *characteristic equation* is $(\lambda-3)(\lambda-3)=0,\quad \lambda^2-6\lambda+9=0$

➔the differential equation is $Y(t)''-6Y'(t)+9Y(t)=0,\quad ==> x^2y''-5xy'+9y=0$

e) $c_1x^3\cos(\ln(x))+c_2x^3\sin(\ln(x))$ (2 scores)

➔ $x=e^t,\quad t=\ln(x),\quad ==> c_1e^{3t}\cos(t)+c_2e^{3t}\sin(t)$

➔ the roots of the *characteristic equation* are $\lambda_1=3+i,\quad \lambda_2=3-i$

➔ the *characteristic equation* is $(\lambda-3-i)(\lambda-3+i)=0,\quad \lambda^2-6\lambda+10=0$

➔the differential equation is $Y(t)'' - 6Y'(t) + 10Y(t) = 0$, $\Longrightarrow x^2y'' - 5xy' + 10y = 0$

4) Consider $y'' + 2y' + y = -3e^{-x} + 8xe^{-x} + 1$

a) find y_1, y_2 to form a fundamental set of solutions of the homogeneous equation $y'' + 2y' + y = 0$ (5 scores)

b) show that y_1, y_2 are linearly independent (5 scores)

c) find the general solution of $y'' + 2y' + y = 0$ (2 scores)

d) find the particular solution of $y'' + 2y' + y = -3e^{-x} + 8xe^{-x} + 1$ (10 scores)

e) write the general solution of $y'' + 2y' + y = -3e^{-x} + 8xe^{-x} + 1$ (3 scores)

a) $\lambda^2 + 2\lambda + 1 = 0$, $\lambda = -1, -1$ ➔ $y_1 = e^{-x}$, $y_2 = xe^{-x}$

b) $W(x) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x} \neq 0$

c) $y_h = c_1e^{-x} + c_2xe^{-x}$

d)Let $y_p = (ax+b)x^2e^{-x} + c$ ➔ $y_p = \frac{4}{3}x^3e^{-x} - \frac{3}{2}x^2e^{-x} + 1$

e) $\varphi = c_1e^{-x} + c_2xe^{-x} + \frac{4}{3}x^3e^{-x} - \frac{3}{2}x^2e^{-x} + 1$

5) Consider the initial value problem $y'' + y = \sec^3(x)$; $y(0) = 4$, $y'(0) = 2$

a) find y_1, y_2 to form a fundamental set of solutions of the homogeneous equation $y'' + y = 0$ (5 scores)

b) find the particular solution of $y'' + y = \sec^3(x)$ (10 scores)

c) find the particular solution of $y'' + y = \sec^3(x)$; $y(0) = 4$, $y'(0) = 2$ (5 scores)

➔ $\lambda^2 + 1 = 0$, $\lambda = i$, $-i$ ➔ $y_h = c_1\cos(x) + c_2\sin(x)$

$y_p = u(x)y_1 + v(x)y_2 = u\cos(x) + v\sin(x)$

$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$

$u'(x) = -\frac{y_2 f(x)}{W(x)} = -\sin(x)\sec^3(x)$, $\Longrightarrow u = -\frac{1}{2\cos^2(x)}$

$v'(x) = \frac{y_1 f(x)}{W(x)} = \cos(x)\sec^3(x) = \sec^2(x)$, $\Longrightarrow v = \tan(x)$

$$y_p = u\cos(x) + v\sin(x) = -\frac{1}{2\cos(x)} + \tan(x)\sin(x) = \frac{\tan(x)\sin(x)}{2}$$

➔ $$y = c_1\cos(x) + c_2\sin(x) + \frac{\tan(x)\sin(x)}{2}$$

$$y(0) = c_1 = 4, \quad y'(0) = c_2 = 2$$

$$y = 4\cos(x) + 2\sin(x) + \frac{\tan(x)\sin(x)}{2}$$

6) Solve $(x^2+1)y'' - 2xy' + 2y = (x^2+1)^2$ with $y_1(x) = 1 - x^2$ (10 scores)

➔ $y_2(x) = x$

$$y_p = uy_1 + vy_2$$

$$f = x^2 + 1$$

$$W = \begin{vmatrix} 1-x^2 & x \\ -2x & 1 \end{vmatrix} = 1 - x^2 + 2x^2 = 1 + x^2$$

➔ $$u' = -\frac{y_2 f}{W} = -\frac{x(x^2+1)}{1+x^2} = -x, \quad u = -\frac{x^2}{2}$$

$$v' = \frac{y_1 f}{W} = \frac{(1-x^2)(x^2+1)}{1+x^2} = 1 - x^2, \quad v = x - \frac{x^3}{3}$$

$$y_p = \frac{x^2(x^2-1)}{2} + x^2 - \frac{x^4}{3}$$

$$\varphi = c_1(1-x^2) + c_2 x + \frac{x^2}{2} + \frac{x^4}{6}$$

7) Find the first five terms of the power series solution of $y'' + xy' - y = e^{3x}$ (15 scores)

See Example 4.7, page 173 in the textbook.