工程數學(一)　　補充資料 1

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Definition

The derivative of a function *f* is denoted as *f'* and defined as following:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$D_x f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example

If $f(x) = x^2$, find $f'(x)$ and $f'(2)$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

$$f'(2) = 2 \times 2 = 4$$

$$D_x\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

$$\begin{aligned} D_x\left[f(x)g(x)\right] &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x)\frac{f(x+h) - f(x)}{h} \\ &= f(x)g(x)' + g(x)f'(x) \end{aligned}$$

Example

Find $D_x(x^3 - x)(x^3 + x)$

$$f(x) = x^3 - x, \quad g(x) = x^3 + x$$

$$D_x(x^3 - x)(x^3 + x) = (x^3 - x)(3x^2 + 1) + (x^3 + x)(3x^2 - 1)$$

$$= 6x^5 - 2x$$

$$D_x \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{hg(x+h)g(x)}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x) - [g(x+h) - g(x)]f(x)}{hg(x+h)g(x)}$$

$$= \lim_{h \to 0} \frac{f'(x)g(x) - g'(x)f(x)}{g(x+h)g(x)}$$

$$= \lim_{h \to 0} \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

Example

Find $D_x \dfrac{x^2}{x^2 - 4}$

$$f(x) = x^2, g(x) = x^2 - 4$$

$$D_x \frac{x^2}{x^2 - 4} = \frac{D_x(x^2)(x^2 - 4) - D_x(x^2 - 4)x^2}{(x^2 - 4)^2}$$

$$= \frac{2x(x^2 - 4) - (2x)x^2}{(x^2 - 4)^2}$$

$$= \frac{-8x}{(x^2 - 4)^2}$$

The Chain Rule

$$D_x f(g(x)) = f'(g(x))g'(x)$$

Set $F(x) = f(g(x))$

$$F'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x))g'(x)$$

$f' = Df$ first derivative of f

$f'' = (f')' = D^2 f$ second derivative of f

$f''' = (f'')' = D^3 f$ third derivative of f

$f^{[4]} = (f''')' = D^4 f$ fourth derivative of f

$? x$ (read delta x): a change of $? x$ in x
$y = f(x)$

$$? y = f(x + ? x) - f(x)$$

$$\frac{dy}{dx} = f'(x) = \lim_{? x \to 0} \frac{? y}{? x}$$

$$f'(x) = \frac{dy}{dx}, \quad f''(x) = \frac{d^2 y}{dx^2}, \quad f'''(x) = \frac{d^3 y}{dx^3}, \ldots, \quad f^{[n]}(x) = \frac{d^n y}{dx^n}$$

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}, \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$y = f(z)$ and $z = g(x)$ so that $y = f(g(x))$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad \frac{dy}{dz} = f'(g(x)), \quad \frac{dz}{dx} = g'(x)$$

differential: dx, dy
$y = f(x)$

$$dy = f'(x)dx = \left(\frac{dy}{dx}\right)dx$$

$$D_x \int f(x)dx = f(x)$$

$$\int D_x f(x)dx = f(x) + C$$

Definition

A function is called 1-1 (one-to-one) function if and only if $f(x_1) \neq f(x_2)$ for any two differential elements x_1 and x_2 of the domain of f.

If $\{(x, y) \mid y = f(x)\}$ is a 1-1 function, then its inverse, designated by f^{-1} is

$\{(y, x) \mid y = f(x)\}$.

$y = f(x) \quad x = f^{-1}(y)$

If f and g are inverse functions which are differentiable, then

$g'(x) = \dfrac{1}{f'(g(x))}$, whenever $f'(g(x)) \neq 0$.

$f(g(x)) = x$

By implicit differentiation

$D_x f(g(x)) = D_x x$

by the chain rule

$f'(g(x))g'(x) = 1$ and

$$g'(x) = \frac{1}{f'(g(x))}$$

Definition

For any positive number a, $a \neq 1$, the ***exponential function*** with base a is defined by

$f(x) = a^x$, domain $f = (-\infty, \infty)$

$a^x a^y = a^{x+y}$

$\frac{a^x}{a^y} a^{x-y}$

$(a^x)^y = a^{xy}$

set $f(x) = a^x$

By definition, for any number x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \left(\frac{a^h - 1}{h} \right)$$

$$f'(x) = a^x \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right)$$

$D_x a^x = k \cdot a^x$ $\qquad$ $k = \lim_{h \to 0} \frac{a^h - 1}{h}$

The inverse of the exponential function $f(x) = a^x$ is called the ***logarithm*** to the base a, abbreviated $\log_a$ domain $\log_a = (0, \infty)$

Thus $\log_a x = y$ if and only if $a^y = x$

$$D_x \log_a x = \frac{1}{kx} \text{ for all } x > 0$$

$$\int_1^x \frac{1}{kx}\, dt = \log_a t \,|_1^x = \log_a x - \log_a 1$$

$$\log_a x = \frac{1}{k}\int_1^x \frac{1}{t}\, dt \qquad k = \lim_{h \to 0} \frac{a^h - 1}{h}$$

$a = 2$, $k \cong 0.693$
$a = 3$, $k \cong 1.099$

The Natural Logarithm

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{domain } \ln = (0, \infty)$$

$$\ln x = \log_e x, \quad e \cong 2.71828 \qquad e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

$$D_x \ln x = \frac{1}{x}$$

$$D_x^2 \ln x = -\frac{1}{x^2}$$

$$\ln(ab) = \ln a + \ln b$$

$$\ln \frac{a}{b} = \ln a - \ln b$$

$$\ln a^r = r \ln a$$

The ***natural exponential function***

$$\exp x = e^x \quad \text{domain } \exp = (-\infty, \infty)$$

Definition

The *hyperbolic sine* designated by sinh

$$\sinh = \frac{e^x - e^{-x}}{2}, \text{domain } \sinh = (-\infty, \infty)$$

The *hyperbolic cosine* designated by cosh

$$\cosh = \frac{e^x + e^{-x}}{2}, \text{domain } \cosh = (-\infty, \infty)$$

$$e^x = \sinh x + \cosh x$$

Periodic function

$$\sin(x+2p)=\sin x,\quad \cos(x+2p)=\cos x$$
$$\sin(x+h)=\sin x\cos h+\cos x\sin h$$

$$D_x\sin(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$
$$=(\sin x)\lim_{h\to 0}\frac{\cos h-1}{h}+(\cos x)\lim_{h\to 0}\frac{\sin h}{h}$$
$$\lim_{h\to 0}\frac{\cos h-1}{h}=0,\quad \lim_{h\to 0}\frac{\sin h}{h}=1$$
$$=\cos x$$

$$D_x\cos(x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$=(\cos x)\lim_{h\to 0}\frac{\cos h-1}{h}-(\sin x)\lim_{h\to 0}\frac{\sin h}{h}$$
$$=(\cos x)\cdot 0-(\sin x)\cdot 1$$
$$=-\sin x$$

$$\int \sin x\ dx=-\cos x+C$$

$$\int \cos x\ dx=\sin x+C$$

<u>**Integration by parts**</u>

$$D_x[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)$$

$$f(x)g(x)=\int f(x)g'(x)\ dx+\int g(x)f'(x)\ dx$$

$$\int f(x)g'(x)\ dx=f(x)g(x)-\int g(x)f'(x)\ dx$$