

1)

$$\begin{aligned} \mathbb{L}[t](s) = F(s) &= \int_0^{\infty} e^{-st} t dt = \lim_{k \rightarrow \infty} \left[-\frac{te^{-st}}{s} \Big|_0^k - \int_0^k -\frac{e^{-st}}{s} dt \right] \\ &= \lim_{k \rightarrow \infty} \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^k = \lim_{k \rightarrow \infty} \left[-\frac{ke^{-sk}}{s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2} \right] = \frac{1}{s^2} \end{aligned}$$

$$\begin{aligned} \mathbb{L}[t^n](s) = F(s) &= \int_0^{\infty} e^{-st} t^n dt = \lim_{k \rightarrow \infty} \left[-\frac{t^n e^{-st}}{s} \Big|_0^k + \frac{n}{s} \int_0^k -e^{-st} t^{n-1} dt \right] \\ &= \frac{n}{s} \int_0^{\infty} e^{-st} t^{n-1} dt \end{aligned}$$

$$\rightarrow \mathbb{L}[t^n](s) = F(s) = \frac{n}{s} \int_0^{\infty} e^{-st} t^{n-1} dt = \frac{n!}{s^{n+1}}$$

2)

$$\mathbb{L}[f'](s) = sF(s) - f(0)$$

$$\text{Set } f = t \rightarrow f' = 1, \quad f(0) = 0$$

$$\rightarrow \mathbb{L}[1](s) = \frac{1}{s} = sF(s) - f(0) = sF(s)$$

$$\rightarrow \mathbb{L}[t](s) = F(s) = \frac{1}{s^2}$$

$$\mathbb{L}[f''](s) = s^2 F(s) - sf(0) - f'(0)$$

$$\text{Set } f = t^2 \rightarrow f' = 2t, \quad f'' = 2, \quad f(0) = 0$$

$$\rightarrow \mathbb{L}[2](s) = \frac{2}{s} = s^2 F(s) - sf(0) - f'(0) = s^2 F(s)$$

$$\rightarrow \mathbb{L}[t^2](s) = F(s) = \frac{2}{s^3}$$

3)

$$\text{Set } f = te^{at} \rightarrow f' = e^{at} + ate^{at}, \quad f(0) = 0$$

$$\mathbb{L}[f'](s) = sF(s) - f(0)$$

$$\rightarrow \mathbb{L}[e^{at} + ate^{at}](s) = sF(s) - f(0)$$

$$\rightarrow \mathbb{L}[e^{at} + ate^{at}](s) = \mathbb{L}[e^{at}](s) + \mathbb{L}[ate^{at}](s) = s\mathbb{L}[te^{at}](s)$$

$$\rightarrow \mathbb{L}[e^{at}](s) = (s-a)\mathbb{L}[te^{at}](s)$$

$$\rightarrow \mathbb{L}[te^{at}](s) = \frac{1}{s-a} \quad \mathbb{L}[e^{at}](s) = \frac{1}{(s-a)^2} \quad (\mathbb{L}[e^{at}](s) = F(s) = \frac{1}{s-a})$$

4)

$$\mathcal{L}[e^{at}](s) = F(s) = \frac{1}{s-a}$$

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$\cosh(at) = \frac{e^{at} + e^{-at}}{2}$$

$$\rightarrow \mathcal{L}[\sinh(at)] = \frac{\frac{1}{s-a} - \frac{1}{s+a}}{2} = \frac{a}{s^2 - a^2}$$

$$\rightarrow \mathcal{L}[\cosh(at)] = \frac{\frac{1}{s-a} + \frac{1}{s+a}}{2} = \frac{s}{s^2 - a^2}$$

5)

$$\rightarrow \mathcal{L}[e^{iat}](s) = F(s) = \frac{1}{s-ia}$$

$$e^{iat} = \cos(at) + i \sin(at) \quad \rightarrow \cos(at) = \frac{e^{iat} + e^{-iat}}{2}, \quad \sin(at) = \frac{e^{iat} - e^{-iat}}{2i}$$

$$e^{-iat} = \cos(at) - i \sin(at)$$

$$\rightarrow \mathcal{L}[\sin(at)](s) = F(s) = \frac{1}{2i} \left[\frac{1}{s-ia} - \frac{1}{s+ia} \right] = \frac{1}{2i} \frac{2ia}{s^2 + a^2} = \frac{a}{s^2 + a^2}$$

$$\rightarrow \mathcal{L}[\cos(at)](s) = F(s) = \frac{1}{2} \left[\frac{1}{s-ia} + \frac{1}{s+ia} \right] = \frac{1}{2} \frac{2s}{s^2 + a^2} = \frac{s}{s^2 + a^2}$$

6)

$$\text{Set } f = te^{iwt} \rightarrow f' = e^{iwt} + i w t e^{iwt}, \quad f(0) = 0$$

$$\mathcal{L}[f'](s) = sF(s) - f(0)$$

$$\rightarrow \mathcal{L}[e^{iwt} + i w t e^{iwt}](s) = sF(s) - f(0)$$

$$\rightarrow \mathcal{L}[e^{iwt} + i w t e^{iwt}](s) = \mathcal{L}[e^{iwt}](s) + \mathcal{L}[i w t e^{iwt}](s) = s \mathcal{L}[te^{iwt}](s)$$

$$\rightarrow \mathcal{L}[e^{iwt} + i w t e^{iwt}](s) = \mathcal{L}[e^{iwt}](s) + \mathcal{L}[i w t e^{iwt}](s) = s \mathcal{L}[te^{iwt}](s)$$

$$\rightarrow \mathcal{L}[te^{iwt}](s) = \frac{1}{s-iw} \mathcal{L}[e^{iwt}](s) = \frac{1}{(s-iw)^2}$$

$$\rightarrow \mathcal{L}[te^{iwt}](s) = \mathcal{L}[t \cos(wt) + i t \sin(wt)](s) = \frac{1}{(s-iw)^2}$$

$$\rightarrow \mathcal{L}[t \cos(wt)](s) + i \mathcal{L}[t \sin(wt)](s) = \frac{1}{(s-iw)^2} = \frac{1}{s^2 - 2iw - w^2} = \frac{s^2 - w^2 + 2iw}{(s^2 - w^2)^2 - (2iw)^2}$$

$$\rightarrow \mathcal{L}[t \cos(wt)](s) + i \mathcal{L}[t \sin(wt)](s) = \frac{s^2 - w^2 + 2iw}{(s^2 - w^2)^2 - (2iw)^2} = \frac{s^2 - w^2}{(s^2 + w^2)^2} + i \frac{2w}{(s^2 + w^2)^2}$$

$$\rightarrow \mathcal{L}[t \cos(wt)](s) = \frac{s^2 - w^2}{(s^2 + w^2)^2}$$

$$\rightarrow \mathcal{L}[t \sin(wt)](s) = \frac{2w}{(s^2 + w^2)^2}$$

7)

$$\begin{aligned} \mathcal{L}[t^{-\frac{1}{2}}](s) &= F(s) = \int_0^{\infty} e^{-st} t^{-\frac{1}{2}} dt \quad (\text{令 } u=st) \\ &= \int_0^{\infty} e^{-u} \left(\sqrt{\frac{s}{u}}\right) \left(\frac{du}{s}\right) = \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-u} u^{-\frac{1}{2}} du \\ &= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-u} u^{-\frac{1}{2}} du \quad (u=T^2) \\ &= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-T^2} \left(\frac{1}{T}\right) 2T dT = \\ &= \frac{1}{\sqrt{s}} \left(\int_0^{\infty} 2e^{-T^2} dT\right) = \frac{1}{\sqrt{s}} \left(2 \frac{\sqrt{\pi}}{2}\right) = \sqrt{\frac{\pi}{s}} \end{aligned}$$

$$\left(\int_0^{\infty} e^{-x^2} dx = ?\right)$$

$$\Rightarrow I = \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y^2} dy$$

$$\rightarrow I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dy dx = \iint_S e^{-(x^2+y^2)} dA \quad S = \{(x,y) | x \geq 0, y \geq 0\}$$

$$\rightarrow I^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} r e^{-(r^2)} dr d\theta = \frac{\pi}{4}$$

$$\rightarrow I = \frac{\sqrt{\pi}}{2}$$