QUIZ-2^{nd} 日河工 2B

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1) Verify the given function is a solution of the differential equation (10 scores)

$$y' = \frac{y}{x} + 1;\ \ \varphi(x) = x\ln(x) + Cx \ \text{ for all } \ x > 0$$

$$\varphi'(x) = \ln(x) + 1 + C,$$

$$\frac{\varphi(x)}{x} = \frac{x\ln(x) + Cx}{x} = \ln(x) + C$$

➔ $\dfrac{\varphi(x)}{x} + 1 = \ln(x) + C + 1 = \varphi'(x)$

2) Verify by implicit differentiation that the given equation implicitly defines a solution of the differential equation (10 scores)

$$y^2 + xy - 2x^2 - 3x - 2y = C;\ \ y - 4x - 3 + (x + 2y - 2)y' = 0$$

$d(y^2 + xy - 2x^2 - 3x - 2y)/dx = d(C)/dx$

==> $2yy' + y + xy' - 4x - 3 - 2y' = 0$

==> $y - 4x - 3 + xy' + 2yy' - 2y' = 0$

==> $y - 4x - 3 + (x + 2y - 2)y' = 0$

3) Solve the differential equation $3y' = 4x/y^2$ (15 scores)

➔ $3y^2dy = 4xdx$, the differential equation is separable.

By direct integration, $\int 3y^2dy = \int 4xdx$ ➔ $y^3 = 2x^2 + C$

4) Solve the differential equation $y' + y = \sin(x)$ (20 scores)

Compare to the general form of linear DE $y' + p(x)y = q(x)$

➔ $p(x) = 1, \quad q(x) = \sin(x)$, the first-order differential equation is linear.

➔An integrating factor is $e^{\int p(x)dx} = e^{\int dx} = e^{x}$

➔ $y'e^{x} + ye^{x} = e^{x}\sin(x)$, $\frac{d}{dx}(ye^{x}) = e^{x}\sin(x)$

➔ $d(ye^{x}) = \int e^{x}\sin(x)dx$, $ye^{x} = \int e^{x}\sin(x)dx = \frac{1}{2}e^{x}[\sin(x) - \cos(x)] + C$

The general solution is $y = \frac{1}{2}[\sin(x) - \cos(x)] + Ce^{-x}$

Note: **<u>Integration by parts</u>**

$$D_x[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$f(x)g(x) = \int f(x)g'(x)\ dx + \int g(x)f'(x)\ dx$$

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int g(x)f'(x)\ dx$$

5) Solve the differential equation $\frac{2xy}{y-1} - y' = 0$ (15 scores)

➔ $\frac{y-1}{y}dy = 2xdx, y \neq 0$, $\int \left(1 - \frac{1}{y}\right)dy = \int 2xdx$, $y - \ln|y| = x^{2} + C$

The general solution is $y - \ln|y| = x^{2} + C$

But $y = 0$ is still a solution of the DE (try it by substitution), though it cannot be contained in the expression for the general solution for any choice of C.

6) For the differential equation $1 + (3x - e^{-2y})y' = 0$, (a)show that it is not exact, (b) find an integrating factor, (c)find the general solution (20 scores)

a)Compare to $M(x,y) + N(x,y)y' = 0$, ➔ $M = 1$, $N = 3x - e^{-2y}$

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 3$, $\partial M/\partial y \neq \partial N/\partial x$, it is not exact.

b) $\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$, ➔ $\frac{\partial}{\partial x}[\mu(3x - e^{-2y})] = \frac{\partial}{\partial y}(\mu)$

Try to find μ as just a function of y, ➔ $\frac{\partial \mu}{\partial x} = 0$

➔ $\mu \frac{\partial}{\partial x}(3x) = \frac{\partial \mu}{\partial y}$, $\frac{\partial \mu}{\partial y} = 3\mu$, $\frac{d\mu}{\mu} = 3dy$, ➔ $\int \frac{d\mu}{\mu} = \int 3dy$

➔ $\ln|\mu| = 3y$, note: here we set the integration constant $C = 0$. ➔ $\mu = e^{3y}$

c)➔ $e^{3y}\left[1 + (3x - e^{-2y})y'\right] = 0$, $e^{3y} + (3xe^{3y} - e^{y})y' = 0$ (now it is exact)

➔The general solution is $xe^{3y} - e^{y} = C$

7) For a first-order differential equation, $y' = \frac{y}{x+y}$, check if it is homogeneous,

then find its general solution. (20 scores)

See page 40 of the textbook !

8) Consider the Riccati equation, $y' = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}$, with a solution $S(x) = 1$.

Try to get its general solution. *Hint*: Define a new variable z by setting

$y = S(x) + \frac{1}{z}$ (20 scores)

See page 45 of the textbook !