海洋大學河海工程學系九十四學年度 第一學期 ■期中 □小考 考試命題紙

考　試　科　目	開課系級	考試日期	印製份數	答案紙	命題教師	備　　註
工程數學一	二 AB	1 月 19 日		■ 需 □不需	陳桂鴻 呂學育	期末考

學生可帶 □書本 ■計算機 □其他__________ ■皆不可　　共 2 頁, 第 1 頁

1. $A=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}$.

(1) Find all eigenvalues and corresponding eigenvector. 3%

Ans: $\lambda=1,1;\ \underset{\sim}{x}=\{1\quad 1\}^T$

(2) Find generalized eigenvector and obtain the transition matrix P of A. 5%

Ans: $[A-\lambda I]\{x_2\}=\{x_1\},\ \{x_2\}=\{0\quad 1\}^T$ and $P=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

(3) Find P^{-1}. 2%

Ans: $P^{-1}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$

(4) Find the Jordan canonical form of A by using the similar transform ($P^{-1}AP$). 5%

Ans: $D=P^{-1}AP=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

2. $A=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$

(1) Find eigenvectors and write the transition matrix P of A. 3%

Ans: $\lambda_1=-1,\ \lambda_2=3;\ \underset{\sim}{x}_1=\{1\quad 1\}^T,\ \underset{\sim}{x}_2=\{1\quad -1\}^T,\ P=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$

(2) Find P^{-1} by using the orthogonal matrix property. 3%

Ans: $\therefore P^{-1}=P^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$

(3) Find the diagonal form of A by using the similar transform ($P^{-1}AP$). 3%

Ans: $P^{-1}AP=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$

(4) If $f(x)=x^{100}$, find the matrix $f(A)$ by using (a) the method of similar transform (matrix function), 9% (b) Cayley-Hamilton theory. 9%

Ans: (a) $f(A)=PDP^{-1}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}(-1)^{100} & 0\\ 0 & 3^{100}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}(-1)^{100}+3^{100} & (-1)^{100}-3^{100}\\ (-1)^{100}-3^{100} & (-1)^{100}+3^{100}\end{bmatrix}$

(b) $\lambda^2-2\lambda-3=0$, $\lambda^m=c_0\lambda+c_1$, $\begin{cases}(-1)^{100}=-c_0+c_1\\ 3^{100}=3c_0+c_1\end{cases}$,

$$c_0=\frac{-1}{4}[(-1)^{100}-3^{100}],\quad c_1=\frac{3}{4}(-1)^{100}+\frac{1}{4}(3^{100})$$

$$f(A)=A^{100}=c_0A+c_1I=\frac{1}{2}\begin{Bmatrix}(-1)^{100}+3^{100} & (-1)^{100}-3^{100}\\ (-1)^{100}-3^{100} & (-1)^{100}+3^{100}\end{Bmatrix}$$

(5) Find A^{-1} by using (a) adjoint method, 2% (b) Cayley-Hamilton theory. 6%

Ans: (a) $A^{-1}=\frac{-1}{3}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

(b) $\lambda^2-2\lambda-3=0\rightarrow A^2-2A-3I=0\rightarrow I=\frac{1}{3}A^2-\frac{2}{3}A\rightarrow A^{-1}=\frac{1}{3}A-\frac{2}{3}I$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}-\frac{2}{3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{-1}{3}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$$

3. For the given linear system

$$\begin{aligned}-x_1+3x_2 \quad &=0\\ x_1-2x_2+x_3&=1\\ x_2+2x_3&=0\end{aligned}$$

we can rewrite it as a matrix-vector equation $AX=B$

with the matrix $A=\begin{pmatrix}-1 & 3 & 0\\ 1 & -2 & 1\\ 0 & 1 & 2\end{pmatrix}$, the vector $X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

(1) write out the vector B (1%)

Ans: $B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(2) calculate det A (1%)

Ans: det A= $\begin{vmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4+0+0-0-6+1=-1$

(3) is the matrix A nonsingular ? (1%)

Ans: $\det A \neq 0 \rightarrow A$ is nonsingular.

(4) write out the adjoint of the matrix A (2%)

Ans:

$$c_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -5,\ c_{12} = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2,\ c_{13} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$$

$$c_{21} = -\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = -6,\ c_{22} = \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2,\ c_{23} = -\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = 1$$

$$c_{31} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3,\ c_{32} = -\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = 1,\ c_{33} = \begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} = -1$$

$$\text{Adj } A = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^T = \begin{pmatrix} -5 & -6 & 3 \\ -2 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

(5) find the inverse of the matrix A (2%)

Ans: $A^{-1} = \dfrac{1}{\det A} adjA = \begin{pmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

(6) solve the system to give the vector X (2%)

Ans: $X = A^{-1}B = \begin{pmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}$

(7) for the matrix A, what is the maximum number of independent column vectors? (1%)

Ans: $\det A \neq 0 \rightarrow$ the maximum number of independent column vectors is **3**

(8) what is the rank of the matrix A ? (2%)

Ans: the rank of the matrix A is **3**

4. For a given matrix $A=\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

(1) find the eigenvalues (hint: with 1 as an eigenvalue of multiplicity 2) (2%)

Ans: $\det(A-\lambda I)=\begin{vmatrix} -\lambda & -1 & -1 \\ -1 & -\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix}=-\lambda^3-2+3\lambda=0 \rightarrow (\lambda+2)(\lambda-1)^2=0$

$\rightarrow \lambda=-2,1,1$

(2) compute A^m; $m=10$ by using Cayley-Hamilton theorem (9%)

Ans: $\lambda^m=c_0+c_1\lambda+c_2\lambda^2 \rightarrow c_0=\dfrac{8-6m+(-2)^m}{9}$, $c_1=\dfrac{2+3m+(-2)^{m+1}}{9}$, $c_2=\dfrac{-1+3m+(-2)^m}{9}$

By Cayley-Hamilton Theorem $\rightarrow A^m=c_0I+c_1A+c_2A^2$

$$\rightarrow A^m=\frac{1}{9}\begin{pmatrix} 8-6m+(-2)^m & 0 & 0 \\ 0 & 8-6m+(-2)^m & 0 \\ 0 & 0 & 8-6m+(-2)^m \end{pmatrix}$$

$$-\frac{1}{9}\begin{pmatrix} 0 & 2+3m+(-2)^{m+1} & 2+3m+(-2)^{m+1} \\ 2+3m+(-2)^{m+1} & 0 & 2+3m+(-2)^{m+1} \\ 2+3m+(-2)^{m+1} & 2+3m+(-2)^{m+1} & 0 \end{pmatrix}$$

$$+\frac{1}{9}\begin{pmatrix} -2+6m-(-2)^{m+1} & -1+3m+(-2)^m & -1+3m+(-2)^m \\ -1+3m+(-2)^m & -2+6m-(-2)^{m+1} & -1+3m+(-2)^m \\ -1+3m+(-2)^m & -1+3m+(-2)^m & -2+6m-(-2)^{m+1} \end{pmatrix}$$

$$\rightarrow A^m=\frac{1}{3}\begin{pmatrix} 2+(-2)^m & -1+(-2)^m & -1+(-2)^m \\ -1+(-2)^m & 2+(-2)^m & -1+(-2)^m \\ -1+(-2)^m & -1+(-2)^m & 2+(-2)^m \end{pmatrix}$$

$$\rightarrow A^{10}=\frac{1}{3}\begin{pmatrix} 2+(-2)^{10} & -1+(-2)^{10} & -1+(-2)^{10} \\ -1+(-2)^{10} & 2+(-2)^{10} & -1+(-2)^{10} \\ -1+(-2)^{10} & -1+(-2)^{10} & 2+(-2)^{10} \end{pmatrix}$$

$$=\begin{pmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{pmatrix}$$

(3) find a set of three mutually orthogonal eigenvectors (9%)

Ans:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}=0 \rightarrow K_1=\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0 \to K_2 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix}$$

$$\to K_2 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{(not unique)}$$

For $\lambda = 1$ again, $\to \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0 \to K_3 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix}$

$$\to K_2{}^T K_3 = 0 \to 2\alpha + \beta = 0 \quad \to K_3 = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

(4) use these vectors obtained in (3) to construct an orthogonal matrix that diagonalizes the matrix A (3%)

Ans:

$$\to P = \begin{pmatrix} \frac{K_1}{\|K_1\|} & \frac{K_2}{\|K_2\|} & \frac{K_3}{\|K_3\|} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

(5) compute A^m; $m = 10$ by diagonalizing the matrix A (5%)

Ans: $\to P^{-1}AP = D \to \left(P^{-1}AP\right)\left(P^{-1}AP\right)...\left(P^{-1}AP\right) = D^m \to P^{-1}A^m P = D^m$

$$\to A^{10} = PD^{10}P^{-1} = PD^{10}P^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} (-2)^{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{pmatrix}$$

5. For a given conic section of the form $2xy = 1$

(1) write the equation as the matrix product $X^T AX = 1$, with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ (2%)

Ans: $\rightarrow \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$

(2) eliminate the xy-term by means of an orthogonal matrix and diagonalization (8%)

Ans: $\rightarrow \lambda = 1, -1$

For $\lambda = 1$ $\quad \rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \rightarrow K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

For $\lambda = -1$ $\quad \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \quad \rightarrow K_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

so that $P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, $\quad P^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

If we define the change of variables $X = PX'$, with $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$\rightarrow X^T AX = (X')^T P^T APX' = (X')^T (P^T AP) X' = (X')^T DX'$

$\rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 1$ or $x'^2 - y'^2 = 1$