Clairaut's Equation

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The ordinary differential equation

$$y = x\frac{dy}{dx} + f(\frac{dy}{dx}) \tag{1}$$

is called Clairaut's equation with f a given differentiable real function.

To solve the equation we use an auxiliary parameter $p = \frac{dy}{dx}$ and rewrite Eq. (1) as

$$y = px + f(p) \tag{2}$$

Differentiating this equation yields

$$\frac{dy}{dx} = p = x\frac{dp}{dx} + p + f^{'}(p)\frac{dp}{dx} \tag{3}$$

or

$$\left[x + f^{'}(p)\right]\frac{dp}{dx} = 0 \tag{4}$$

The equation now gives the alternatives

$$\frac{dp}{dx} = 0 \tag{5}$$

and

$$x + f^{'}(p) = 0 \tag{6}$$

Integrating Eq.(5) we obtain $p = c$ with c a constant, and substituting this back into Eq. (1) gives the general solution

$$y = cx + f(c) \tag{7}$$

Obviously, Eq. (7) represents a family of straight lines.

If Eq. (6) allows to solve p in terms of x, $p = p(x)$, we can write Eq. (1) as

$$y = xp(x) + f(p(x)) \tag{8}$$

which is satisfying Eq. (1). The solution (8) may not be obtained from Eq. (7) using any value of c. Thus, it is a singular solution and may be obtained by eliminating the parameter p from the equations

$$y = px + f(p), \tag{9}$$

$$x + f'(p) = 0 \tag{10}$$

And the singular solution presents the envelope of the family (7).

Example:

Consider the equation of Clairaut

$$y = x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

➔ $f(\frac{dy}{dx}) = -\left(\frac{dy}{dx}\right)^2$ ➔ $f(p) = -(p)^2$ with $p = \frac{dy}{dx}$ ➔ $f'(p) = -2p$

By Eq. (7), the **general solution**

➔ $y = cx + f(c)$ ➔ $y = cx - c^2$

By Eqs. (9), (10), the **singular solution**

➔ $x + f'(p) = 0$ ➔ $x - 2p = 0$

➔ $p = \frac{dy}{dx} = \frac{x}{2}$

➔ $y = x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 = x\frac{x}{2} - \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$