For a given nonexact differential equation
(針對一非正合微分方程式)

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

Let's find an integrating factor $\mu(x,y)$ such that $\mu M(x,y)dx + \mu N(x,y)dy = 0$ is exact.
(找出一積分因子 $\mu(x,y)$ ，使得 $\mu M(x,y)dx + \mu N(x,y)dy = 0$ 爲一正合微分方程式。)

According to the criterion for an exact differential equation
(依據正合微分方程式之判別準則)

$$\Rightarrow \frac{\partial(\mu M(x,y))}{\partial y} = \frac{\partial(\mu N(x,y))}{\partial x} \tag{2}$$

$$\Rightarrow \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x} \tag{3}$$

Sometimes equation (3) becomes simple enough to solve if we try μ as a function of just x or just y.
(有時，若 μ 只取爲 x 的函數或只取爲 y 的函數，則式子 3 將變得夠簡單而可被求解)

1)Consider $\mu = \mu(x)$ (當 μ 只取爲 x 的函數)

$\Rightarrow \frac{\partial \mu}{\partial y} = 0$, $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$ (因只是爲 x 的函數，所以對 y 的微分爲零)

$\Rightarrow \mu \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x}$ (式子 3 化簡結果)

$\Rightarrow \frac{1}{N}\frac{\partial M}{\partial y} = \frac{1}{\mu}\frac{d\mu}{dx} + \frac{1}{N}\frac{\partial N}{\partial x}$ (上式移項處理)

$\Rightarrow \frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ (上式移項處理)

$\Rightarrow \frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R(x)$ (承接上式，因左邊顯然爲 x 的函數，右邊自然亦須爲 x 的函數)

$\Rightarrow \int \frac{d\mu}{\mu} = \int R(x)dx$ (承接上式，變數可分離，直接作積分)

$\Rightarrow \ln|\mu| = \int R(x)dx + c_1$ $\Rightarrow |\mu| = e^{c_1} e^{\int R(x)dx}$ $\Rightarrow \mu = \pm e^{c_1} e^{\int R(x)dx} = ce^{\int R(x)dx}$

➔ $\mu = e^{\int R(x)dx}$ (c=1，取最簡單型式之積分因子)

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R(x)$ ➔ $\mu = e^{\int R(x)dx}$ (最左邊前題成立，則 μ 可只取爲 x 的函數)。

2) Consider $\mu = \mu(y)$ (當 μ 只取爲 y 的函數)

➔ $\frac{\partial \mu}{\partial x} = 0$, $\frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$ (因只是爲 y 的函數，所以對 x 的微分爲零)

➔ $\frac{d\mu}{dy} M + \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x}$ (式子 3 化簡結果)

➔ $\frac{d\mu}{dy} M = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ (上式移項處理)

➔ $\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ (上式移項處理)

➔ $\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \tilde{R}(y)$ (承接上式，因左邊顯然爲 y 的函數，右邊自然亦須爲 y 的函數)

➔ $\int \frac{d\mu}{\mu} = \int \tilde{R}(y)dy$ (承接上式，變數可分離，直接作積分)

➔ $\ln|\mu| = \int \tilde{R}(y)dy + c_1$ ➔ $|\mu| = e^{c_1} e^{\int \tilde{R}(y)dy}$ ➔ $\mu = \pm e^{c_1} e^{\int \tilde{R}(y)dy} = c e^{\int \tilde{R}(y)dy}$

➔ $\mu = e^{\int \tilde{R}(y)dy}$ (c=1，取最簡單型式之積分因子)

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \tilde{R}(y)$ ➔ $\mu = e^{\int \tilde{R}(y)dy}$ (最左邊前題成立，則 μ 可只取爲 y 的函數)。