海洋大學河海工程學系九十四學年度 第一學期 ■期中 □小考 考試命題紙

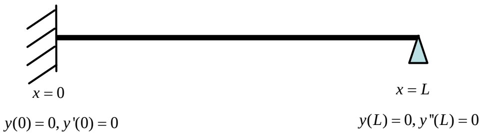
考 試 科 目	開課系級	考試日期	印製份數	答案紙	命題教師	備 註
工程數學一	二 B	12 月 26 日	62	■ 需 □不需	陳桂鴻 呂學育	第三次大考

學生可帶 □書本 ■計算機 □其他__________ □皆不可

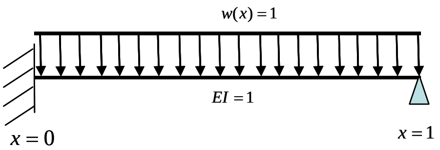
1. A beam with flexural rigidity EI and length L is subject to the load per unit length $w(x)$.

The differential equation of the deflection is $EI\dfrac{d^4y(x)}{dx^4}=w(x)$.

The boundary conditions is shown as



When the beam is embedded at its left end and simply supported at it right end and $EI=1$, $w(x)=1$, L=1, as follows:



Find the deflection of the beam by using

(a) the method of undermined coefficients. (7%)

ANS Let $y_c=e^{\lambda x}$ $\therefore y_c=c_1+c_2x+c_3x^2+c_4x^3$

Let $y_p=ax^4$, $y_p'=4ax^3$, $y_p''=12ax^2$, $y_p'''=24ax$, $y_p^{(4)}=24a$ $\therefore a=\dfrac{w(x)}{24EI}$

$y=c_1+c_2x+c_3x^2+c_4x^3+\dfrac{w(x)}{24EI}x^4$, B.C. $y(0)=0$, $y'(0)=0$, $y(L)=0$, $y''(L)=0$

$y(0)=0\rightarrow c_1=0$, $y'(0)=0\rightarrow c_2=0$,

$y(L)=0$, $y''(L)=0\rightarrow c_3=\dfrac{3w(x)}{48EI}L^2$, $c_4=\dfrac{-5w(x)}{48EI}L$

$\therefore y=\dfrac{1}{16}x^2-\dfrac{5}{48}x^3+\dfrac{1}{24}x^4$

(b) Taylor series expansion method. (9%)

ANS $y(x)=y(0)+\dfrac{y'(0)}{1!}x+\dfrac{y''(0)}{2!}x^2+\dfrac{y'''(0)}{3!}x^3+\dfrac{y^{(4)}(0)}{4!}x^4+\dfrac{y^{(5)}(0)}{5!}x^5+\cdots$

Let $y(0)=a$, $y'(0)=b$, $y''(0)=c$, $y'''(0)=d$

$EIy^{(4)}=w(x)\rightarrow y^{(4)}(0)=\dfrac{w(x)}{EI}$, $EIy^{(5)}=0\rightarrow y^{(5)}(0)=0$

$$y(x) = a + bx + \frac{c}{2}x^2 + \frac{d}{6}x^3 + \frac{w(x)}{24EI}x^4, \quad y'(x) = b + cx + \frac{d}{2}x^2 + \frac{w(x)}{6EI}x^3, \quad y''(x) = c + dx + \frac{w(x)}{2EI}x^2$$

$$y(0) = 0 \to a = 0, \quad y'(0) = 0 \to b = 0,$$

$$y(L) = 0, \quad y''(L) = 0 \to c = \frac{w(x)}{8EI}L^2, d = \frac{-5w(x)}{8EI}L$$

$$\therefore y(x) = \frac{1}{16}x^2 - \frac{5}{48}x^3 + \frac{1}{24}x^4$$

(c) power series with recurrence relation. (9%)

ANS Let $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$, $y''' = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3}$,

$$y^{(4)} = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)c_n x^{n-4}$$

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)c_n x^{n-4} = \sum_{k=0}^{\infty} (k+4)(k+3)(k+2)(k+1)c_{k+4} x^k = \frac{w(x)}{EI}$$

$$k = 0, \quad c_4 = \frac{w(x)}{24EI}, \qquad k = 1, \quad c_5 = 0, \qquad \therefore y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \frac{w(x)}{24EI}x^4$$

$$y(0) = 0 \to c_1 = 0, \quad y'(0) = 0 \to c_2 = 0,$$

$$y(L) = 0, \quad y''(L) = 0 \to c_3 = \frac{3w(x)}{48EI}L^2, c_4 = \frac{-5w(x)}{48EI}L$$

$$\therefore y = \frac{1}{16}x^2 - \frac{5}{48}x^3 + \frac{1}{24}x^4$$

2. Given differential equation as follows:

$$xy'' + y' + xy = 0$$

(a) Determine the singular points of the given D.E. and classify (prove) each singular point as regular or irregular. (5%)

ANS $y'' + \frac{1}{x}y' + y = 0$, $x = 0$ is singular point

$x(\frac{1}{x}) = 1$, $x^2(1) = x^2$ $\quad \therefore x = 0$ is regular singular point

(b) Use the method of Frobenius to obtain the general solution. (20%)

ANS Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$

$$xy'' + y' + xy$$

$$= x\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} + x\sum_{n=0}^{\infty} c_n x^{n+r}$$

$$=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}+\sum_{n=0}^{\infty}c_n x^{n+r+1}$$

$$=x^r\left\{r^2c_0x^{-1}+(r+1)^2c_1+\sum_{k=1}^{\infty}\left[(k+r+1)(k+r+1)c_{k+1}+c_{k-1}\right]x^k\right\}=0$$

Let $c_1=0$, $c_0\neq 0$, $\therefore r=0,0$, and $c_{k+1}=-\dfrac{c_{k-1}}{(k+r+1)^2}=-\dfrac{c_{k-1}}{(k+1)^2}$ $\quad k=1,2,3,\cdots$

$r=0$, $c_{k+1}=\dfrac{-c_{k-1}}{(k+1)^2}$, $\quad c_2=\dfrac{-c_0}{2^2}$, $c_3=0$, $c_4=\dfrac{c_0}{2^2\cdot 4^2}$, $c_5=0$, $c_6=\dfrac{-c_0}{2^2\cdot 4^2\cdot 6^2}$, $c_7=0$,

$c_8=\dfrac{c_0}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}$,

$$\therefore y_1(x)=c_0-\frac{c_0}{2^2}x^2+\frac{c_0}{2^2\cdot 4^2}x^4-\frac{c_0}{2^2\cdot 4^2\cdot 6^2}x^6+\frac{c_0}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}x^8+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}$$

Assume $y_2(x)=y_1\ln|x|+\sum_{n=1}^{\infty}b_nx^n$

$$y_2'(x)=y_1'(x)\ln|x|+\frac{y_1(x)}{x}+\sum_{n=1}^{\infty}nb_nx^{n-1}$$

$$y_2''(x)=y_1''(x)\ln|x|+\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}+\sum_{n=2}^{\infty}n(n-1)b_nx^{n-2}$$

$xy''+y'+xy$

$$=y_1''(x)x\ln|x|+2y_1'(x)-\frac{y_1(x)}{x}+\sum_{n=2}^{\infty}n(n-1)b_nx^{n-1}+y_1'(x)\ln|x|+\frac{y_1(x)}{x}$$

$$+\sum_{n=1}^{\infty}nb_nx^{n-1}+y_1\ln|x|+\sum_{n=1}^{\infty}b_nx^{n+1}$$

$$=2y_1'(x)+\sum_{n=2}^{\infty}n(n-1)b_nx^{n-1}+\sum_{n=1}^{\infty}nb_nx^{n-1}+\sum_{n=1}^{\infty}b_nx^{n+1}$$

$$=2y_1'(x)+b_1+4b_2x+\sum_{k=2}^{\infty}k(k+1)b_{k+1}x^k+\sum_{k=2}^{\infty}(k+1)b_{k+1}x^k+\sum_{k=2}^{\infty}b_{k-1}x^k$$

$$=b_1+4b_2x+\sum_{n=1}^{\infty}\frac{n(-1)^n}{2^{2n-2}(n!)^2}x^{2n-1}+\sum_{k=2}^{\infty}x^k[(k+1)^2b_{k+1}+b_{k-1}]$$

$$=b_1+4b_2x+\sum_{k=0}^{\infty}\frac{(k+1)(-1)^{k+1}}{2^{2k}[(k+1)!]^2}x^{2k+1}+\sum_{k=1}^{\infty}[(2k+2)^2b_{2k+2}+b_{2k}]x^{2k+1}+\sum_{k=1}^{\infty}[(2k+1)^2b_{2k+1}+b_{2k-1}]x^{2k}$$

$$=b_1+(4b_2-1)x+\sum_{k=1}^{\infty}\frac{(k+1)(-1)^{k+1}}{2^{2k}[(k+1)!]^2}x^{2k+1}+\sum_{k=1}^{\infty}[(2k+2)^2b_{2k+2}+b_{2k}]x^{2k+1}+\sum_{k=1}^{\infty}[(2k+1)^2b_{2k+1}+b_{2k-1}]x^{2k}$$

$$=b_1+(4b_2-1)x+\sum_{k=1}^{\infty}[\frac{(k+1)(-1)^{k+1}}{2^{2k}[(k+1)!]^2}+(2k+2)^2b_{2k+2}+b_{2k}]x^{2k+1}+\sum_{k=1}^{\infty}[(2k+1)^2b_{2k+1}+b_{2k-1}]x^{2k}=0$$

Set $b_1=0$, $b_0\neq 0$, $\therefore b_2=\dfrac{1}{4}$

$$\frac{(k+1)(-1)^{k+1}}{2^{2k}[(k+1)!]^2}+(2k+2)^2b_{2k+2}+b_{2k}=0 \quad\longrightarrow ①$$

$(2k+1)^2 b_{2k+1} + b_{2k-1} = 0 \longrightarrow$ ②

According to the ②, so we can know $b_3 = b_5 = b_7 = \cdots = 0$

According to the ①, we get $b_4 = -\frac{3}{128}$, $b_6 = \frac{11}{13824}$, $b_8 = -\frac{25}{1769472}$, $\cdots$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} \ln|x| + (\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \frac{25}{1769472}x^8 + \cdots)$$

$$\therefore y(x) = C_1 y_1(x) + C_2 y_2(x)$$

3. (1) Consider $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n n}$

(a) find the radius of convergence. (3%)

ANS 3

(b) find the interval of convergence. (3%)

ANS $(-1,5)$

(2) Consider $(x^3 - 2x^2 - 3x)^2 y'' + x(x-3)^2 y' - (x+1)y = 0$

(a) determine the singular points. (3%)

ANS $x = 0, 3, -1$

(b) classify each singular points as regular or irregular. (3%)

ANS $x = 0, 3 \to regular$, $x = -1 \to irregular$

(c) without solving the general solution, find the indicial roots about $x = 0$. (3%)

ANS $r = 0, 1$

4. Use the method of Frobenius to find the general solution of $x(2-x)y'' - 2(x-1)y' + 2y = 0$. (20%)

ANS Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$

$$x(2-x)y'' - 2(x-1)y' + 2y$$

$$= 2(r-1+r)c_0 x^{r-1} + \sum_{n=1}^{\infty} 2c_n (n+r-1)(n+r) x^{n+r-1} - \sum_{n=0}^{\infty} c_n (n+r-1)(n+r) x^{n+r}$$

$$- \sum_{n=0}^{\infty} 2c_n (n+r) x^{n+r} + \sum_{n=1}^{\infty} 2c_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} 2c_n x^{n+r}$$

$$= 4r^2 c_0 x^{r-1} + \sum_{k=r}^{\infty} [2c_{k-r+1}(k+1)^2 - c_{k-r}(k+2)(k-1)]$$

$\rightarrow 4rc_0x^{r-1}=0$, $2c_{k-r+1}(k+1)^2-c_{k-r}(k+2)(k-1)=0$

$\rightarrow r=0$, $c_0 \neq 0$, $c_{k+1}=\dfrac{(k+2)(k-1)}{2(k+1)^2}c_k$, $k=0,1,2,3,\cdots$

$k=0 \rightarrow c_1=-c_0$

$k=1 \rightarrow c_2=0$

$k=2 \rightarrow c_3=0$

$\vdots$

$y_1=c_0(1-x)$ set $c_0=1 \rightarrow y_1=1-x$

$$y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx=(1-x)\int \frac{e^{-\int \frac{-2(x-1)}{x(2-x)}dx}}{(1-x)^2}dx=\frac{1}{2}(1-x)\ln(\frac{x-2}{x})-1$$

$\therefore y=C_1y_1+C_2y_2$

5. Consider $x^2y''+xy'+(\lambda^2x^2-4)y=0$

(1) for $\lambda=0$,

(a) is it a Cauchy-Euler equation? (2%)

ANS Yes.

(b) write out its general solution. (3%)

ANS Let $y=x^m$, $m^2=4$, $\therefore y=c_1x^2+c_2x^{-2}$.

(2) for $\lambda=1$,

(a) is it called the Bessel's equation? (2%)

ANS Yes.

(b) without solving, write out its general solution. (in terms of Bessel Functions) (3%)

ANS $y=c_1J_2(x)+c_2Y_2(x)$.

(3) for $\lambda \neq 0,\ 1$,

(a) is it called the parametric Bessel's equation? (2%)

ANS Yes.

(b) without solving, write out its general solution. (in terms of Bessel Functions) (3%)

ANS $y=c_1J_2(\lambda x)+c_2Y_2(\lambda x)$.