

考試科目	開課系級	考試日期	印製份數	答案紙	命題教師	備註
工程數學(一)	二 A,B	11月18日	111	■需 □不需	陳桂鴻 呂學育	第二次期中考 共1頁

1. (a) Determine the complementary solution, the particular solution and general solution of a nonhomogeneous linear DE $y^{(4)} + y''' = 1 - x^2 e^{-x}$. (10%)

Ans: $\lambda^4 + \lambda^3 = 0$, $\lambda = -1, 0, 0, 0$, $y_c = c_1 e^{-x} + c_2 + c_3 x + c_4 x^2$

Let $y_p = ax^3 + (bx^3 + cx^2 + dx)e^{-x}$

$$y_p''' = 6a + 6be^{-x} - 3(6bx + 2c)e^{-x} + 3(3bx^2 + 2cx + d)e^{-x} - (bx^3 + cx^2 + dx)e^{-x}$$

$$y_p^{(4)} = -24be^{-x} + 6(6bx + 2c)e^{-x} - 4(3bx^2 + 2cx + d)e^{-x} + (bx^3 + cx^2 + dx)e^{-x}$$

$$a = \frac{1}{6}, b = \frac{1}{3}, c = 3, d = 12, \therefore y_p = \frac{1}{6}x^3 + \left(\frac{1}{3}x^3 + 3x^2 + 12x\right)e^{-x}$$

$$y = c_1 e^{-x} + c_2 + c_3 x + c_4 x^2 + \frac{1}{6}x^3 + \left(\frac{1}{3}x^3 + 3x^2 + 12x\right)e^{-x}$$

(b) Find the general solution of $x^4 y'' + x^3 y' + 4x^2 y = 1$, given that $y_1(x) = x^2$ is a solution of the associated homogeneous equation. (5%)

Ans: $y'' + \frac{1}{x}y' + \frac{4}{x^2}y = \frac{1}{x^4}$, $p(x) = \frac{1}{x}$, $e^{-\int p(x)dx} = x^{-1}$, $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = x^2 \int x^{-5} dx = \frac{-1}{4}x^{-2}$

$$\therefore y_c = c_1 x^2 - c_2 \frac{1}{4} x^{-2}$$

Let $y_p = u_1 x^2 + u_2 \frac{1}{4} x^{-2}$, $\omega = -x^{-1}$, $u_1 = \frac{-1}{16} x^{-4}$, $u_2 = -\ln|x|$, $\therefore y_p = \frac{-1}{16} x^{-2} - \frac{1}{4} x^{-2} \ln|x|$

$$\therefore y = c_1 x^2 - c_2 \frac{1}{4} x^{-2} - \frac{1}{16} x^{-2} - \frac{1}{4} x^{-2} \ln|x| = c_1 x^2 - c_3 x^{-2} - \frac{1}{4} x^{-2} \ln|x|$$

2. Cauchy-Euler equation $x^2 y'' + 10xy' + 8y = x^2$. Find the complementary solution by using

(a) Change of variable method ($t = \ln x$). (10%)

Ans: Let $t = \ln x$, $x = e^t$

$$\therefore x^2 y'' + 10xy' + 8y = x^2 \rightarrow Y''(t) + 9Y'(t) + 8Y(t) = e^{2t}$$

$$\lambda^2 + 9\lambda + 8 = 0, \lambda = -8, -1, \therefore Y_c = c_1 e^{-8t} + c_2 e^{-t}$$

Let $Y_p = u_1 e^{-8t} + u_2 e^{-t}$, $w = 7e^{-9t}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-t} \\ e^{2t} & -e^{-t} \end{vmatrix}}{7e^{-9t}} = \frac{-1}{7} e^{10t}, u_1 = \int u_1' dt = \frac{-1}{70} e^{10t}$$

$$u_2' = \frac{\begin{vmatrix} e^{-8t} & 0 \\ -8e^{-8t} & e^{2t} \end{vmatrix}}{7e^{-9t}} = \frac{1}{7}e^{3t}, \quad u_2 = \int u_2' dt = \frac{1}{21}e^{3t}$$

$$Y_p = \frac{-1}{70}e^{10t}e^{-8t} + \frac{1}{21}e^{3t}e^{-t} = \frac{-1}{70}e^{2t} + \frac{1}{21}e^{2t} = \frac{1}{30}e^{2t}, \quad \therefore y_p = \frac{1}{30}x^2$$

(b) Let $y = x^m$ and look for m . (5%)

Ans: $m^2 + 9m + 8 = 0, \quad m = -8, -1, \quad y_c = c_1x^{-8} + c_2x^{-1}$

Let $y_p = u_1x^{-8} + u_2x^{-1}, \quad w = 7x^{-10}$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}}{7x^{-10}} = \frac{-1}{7}x^9, \quad u_1 = \int u_1' dt = \frac{-1}{70}x^{10}$$

$$u_2' = \frac{\begin{vmatrix} x^{-8} & 0 \\ -8x^{-9} & 1 \end{vmatrix}}{7x^{-10}} = \frac{1}{7}x^2, \quad u_2 = \int u_2' dt = \frac{1}{21}x^3$$

$$y_p = \frac{-1}{70}x^{10}x^{-8} + \frac{1}{21}x^3x^{-1} = \frac{-1}{70}x^2 + \frac{1}{21}x^2 = \frac{1}{30}x^2$$

(c) If we have found one solution, $y_1(x) = x^{-1}$, find y_2 by using reduction order method. (10%)

Ans: $x^2y'' + 10xy' + 8y = x^2 \rightarrow y'' + \frac{10}{x}y' + \frac{8}{x^2}y = 1$

$$p(x) = \frac{10}{x}, \quad e^{-\int p(x)dx} = x^{-10}, \quad y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = x^{-1} \int x^{-8} dx = \frac{-1}{7}x^{-8}$$

3. Solve the singular solution and general solution of the Clairauts equation $y = x \frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$, where

$$f\left(\frac{dy}{dx}\right) = -e^{2y'}. \quad (10\%)$$

Ans: $y = xy' - e^{2y'}, \quad \text{Let } p = y', \quad \therefore y = xp - e^{2p}, \quad y' = p + xp' - 2p'e^{2p}, \quad \therefore p'(x - 2e^{2p}) = 0$

general solution: $\begin{cases} p' = 0 \\ y = xp - e^{2p}, \quad p = c, \quad \therefore y = cx - e^{2c} \end{cases}$

singular solution: $\begin{cases} x - 2e^{2p} = 0 \\ y = xp - e^{2p}, \quad p = \frac{1}{2} \ln \left| \frac{x}{2} \right|, \quad \therefore y = \frac{1}{2} x \ln \left| \frac{x}{2} \right| - \frac{x}{2}. \end{cases}$

4. Consider (a), (b), (c), (d), determine whether the given set of function is linearly dependent or linearly independent on the interval $(0, \infty)$.

(a) $f_1(x) = \cos(2x), \quad f_2(x) = \sin(2x) \quad (3\%)$

Ans: $W(f_1, f_2) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2 \neq 0 \quad \therefore \text{linearly independent}$

(b) $f_1(x) = e^{2x}$, $f_2(x) = e^{-2x}$ (3%)

Ans: $W(f_1, f_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \neq 0 \therefore \text{linearly independent}$

(c) $f_1(x) = x^2$, $f_2(x) = x + 1$ (3%)

Ans: $W(f_1, f_2) = \begin{vmatrix} x^2 & x+1 \\ 2x & 1 \end{vmatrix} = -x^2 - 2x \neq 0 \therefore \text{linearly independent}$

(d) $f_1(x) = 1 + x$, $f_2(x) = -x$, $f_3(x) = -x^2$ (3%)

Ans: $W(f_1, f_2, f_3) = \begin{vmatrix} 1+x & -x & -x^2 \\ 1 & -1 & -2x \\ 0 & 0 & -2 \end{vmatrix} = 2(1+x) - 2x = 2 \neq 0 \therefore \text{linearly independent}$

(e) is the differential equation $(y^2 - 1)dx = xdy$ linear or nonlinear in x ? why? (3%)

Ans: $(y^2 - 1)x' = x$ linear \therefore 無互乘項 無非線性函數

5. Given second-order differential equations on the interval $(-\infty, 0)$

(1) $x^2 y'' - 2xy' + 2y = 5x^3 \cos 3x$

(a) solve the associated homogeneous equation (y_c) (4%)

Ans: Let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$$\rightarrow m(m-1)x^m - 2mx^m + 2x^m = 0 \rightarrow m^2 - 3m + 2 = 0 \rightarrow m = 1, 2$$

$$\rightarrow y_c = c_1 x + c_2 x^2$$

(b) find the particular solution of the nonhomogeneous equation (y_p) (6%)

Ans: Let $y_p = \phi_1(x)x + \phi_2(x)x^2$

$$x^2 y'' - 2xy' + 2y = 5x^3 \cos 3x \rightarrow y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 5x \cos 3x$$

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2, \quad W_1 = \begin{vmatrix} 0 & x^2 \\ 5x \cos 3x & 2x \end{vmatrix} = -5x^3 \cos 3x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 5x \cos 3x \end{vmatrix} = 5x^2 \cos 3x$$

$$\phi_1' = \frac{W_1}{W} = -\frac{5x^3 \cos 3x}{x^2} = -5x \cos 3x \rightarrow \phi_1(x) = -\frac{5}{3} x \sin 3x - \frac{5}{9} \cos 3x$$

$$\phi_2' = \frac{W_2}{W} = \frac{5x^2 \cos 3x}{x^2} = 5 \cos 3x \rightarrow \phi_2(x) = \frac{5}{3} \sin 3x$$

$$\therefore y_p = x\left(-\frac{5}{3} x \sin 3x - \frac{5}{9} \cos 3x\right) + x^2\left(\frac{5}{3} \sin 3x\right)$$

(2) $x^2 y'' - 3xy' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 4$. Find the solution of the differential equation (8%).

Ans: Let $-x = t$

$$t^2 \frac{d^2 y}{dt^2} - 3t \frac{dy}{dt} + 4y = 0 \quad y(1) = 2, \quad y'(1) = -4$$

$$\text{Let } y = t^m \quad y'' = m(m-1)x^{m-2} \quad y' = mx^{m-1}$$

$$\rightarrow (m^2 - mt + 4)x^t = 0 \quad \rightarrow m = 2, 2 \quad \rightarrow y = c_1 t^2 + c_2 t^2 \ln t$$

$$\text{from I.C. } y(1) = 2, \quad y'(1) = -4 \quad \rightarrow y = 2t^2 - 8t^2 \ln t \quad \rightarrow y = 2x^2 - 8x^2 \ln(-x), \quad x < 0$$

6. Solve the given initial-value problem

$$\frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + 4x = \cos(2t); \quad x(0) = 0, \quad x'(0) = 0$$

(a) if $\beta = 0$ find the solution of the differential equation (5%)

$$\text{Ans: } \beta = 0 \quad \rightarrow \frac{d^2 x}{dt^2} + 4x = \cos(2t)$$

1. Let $x = e^{mt}$

$$(m^2 + 4)e^{mt} = 0 \quad \rightarrow m = \pm 2i \quad \rightarrow x_h = c_1 \cos 2t + c_2 \sin 2t$$

2. Let $x_p = \phi_1(t) \cos 2t + \phi_2(t) \sin 2t$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{vmatrix} = 2, \quad W_1 = \begin{vmatrix} 0 & \sin 2t \\ \cos 2t & 2 \cos 2t \end{vmatrix} = -\sin 2t \cos 2t, \quad W_2 = \begin{vmatrix} \cos 2t & 0 \\ -2 \sin 2t & \cos 2t \end{vmatrix} = \cos^2 2t$$

$$\phi_1' = \frac{W_1}{W} = \frac{-\sin 2t \cos 2t}{2} \quad \rightarrow \phi_1 = \frac{\cos 4t}{16}$$

$$\phi_2' = \frac{W_2}{W} = \frac{\cos^2 2t}{2} \quad \rightarrow \phi_2 = \frac{t}{4} + \frac{1}{16} \sin 4t$$

$$x_p = \cos 2t \left(\frac{\cos 4t}{16} \right) + \sin 2t \left(\frac{t}{4} + \frac{1}{16} \sin 4t \right)$$

$$3. \quad x = x_h + x_p = c_1 \cos(2t) + c_2 \sin(2t) + \cos(2t) \left[\frac{\cos(4t)}{16} \right] + \sin(2t) \left[\frac{t}{4} + \frac{1}{16} \sin(4t) \right]$$

$$\text{from I.C. } x(0) = 0, \quad x'(0) = 0 \quad \rightarrow c_1 = \frac{-1}{16}, \quad c_2 = 0$$

$$\rightarrow x = \frac{-1}{16} \sin(2t) + \cos(2t) \left[\frac{\cos(4t)}{16} \right] + \sin(2t) \left[\frac{t}{4} + \frac{1}{16} \sin(4t) \right]$$

$$\rightarrow x = \frac{t \sin(2t)}{4} + \frac{1}{16} \left([1 - 2 \sin^2(2t)] \cos(2t) + 2 \sin^2(2t) \cos(2t) - \cos(2t) \right) \quad \rightarrow x = \frac{t \sin(2t)}{4}$$

(b) if $\beta = 5$ find the solution of the differential equation (7%)

$$\text{Ans: } \beta = 5 \quad \frac{d^2 x}{dt^2} + 5 \frac{dx}{dt} + 4x = \cos(2t)$$

1. Let $x = e^{mt}$

$$(m^2 + 5m + 4) = 0 \quad \rightarrow m = -1, -4 \quad \rightarrow x_h = c_1 e^{-t} + c_2 e^{-4t}$$

2. Let $x_p = A \cos 2t + B \sin 2t \rightarrow x_p' = -2A \sin 2t + 2B \cos 2t \rightarrow x_p'' = -4A \cos 2t - 4B \sin 2t$

$$\rightarrow 10B \cos 2t - 10A \sin 2t = \cos 2t \rightarrow B = \frac{1}{10}, A = 0$$

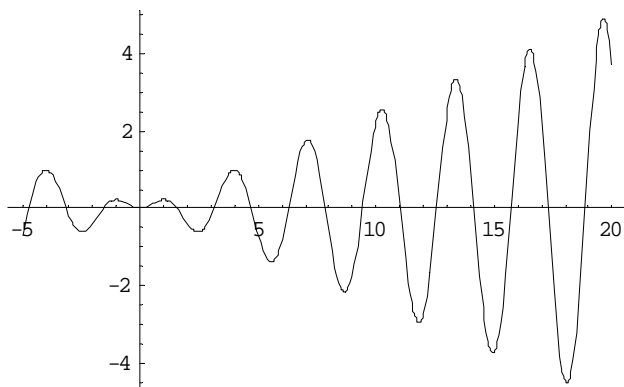
3. $x = x_h + x_p = c_1 e^{-t} + c_2 e^{-4t} + \frac{1}{10} \sin(2t)$

from I.C. $x(0) = 0, x'(0) = 0 \rightarrow x = -\frac{1}{15} e^{-t} + \frac{1}{15} e^{-4t} + \frac{1}{10} \sin(2t)$

(c) if $t \rightarrow \infty$ compare your solutions a) and b) ($x \rightarrow \infty ?$,...) (5%)

Ans: (a) diverge

$$t \rightarrow \infty \Rightarrow \frac{t}{4} \rightarrow \infty, \sin(2t) \rightarrow \text{oscillate}$$



(b) converge

$$t \rightarrow \infty \Rightarrow -\frac{1}{15} e^{-t} \rightarrow 0, \frac{1}{15} e^{-4t} \rightarrow 0, \frac{1}{10} \sin(2t) \rightarrow \text{oscillate}$$

