# Frobenius Method

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## 1)規則奇異點/不規則奇異點之定義

假設 $x = x_0$ 為以下微分方程式

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

之奇異點(singular point)。若微分方程式轉換成以下之標準式：

$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$

則奇異點 $x = x_0$ 為微分方程式(1)之規則奇異點(regular singular point)，假如：

$$p(x) = (x - x_0)P(x) \tag{3}$$

$$q(x) = (x - x_0)^2 Q(x) \tag{4}$$

其中函數 $p(x)$，$q(x)$ 在 $x = x_0$ 皆為解析(analytic)。否則奇異點 $x = x_0$ 為微分方程式(1)之不規則奇異點(irregular singular point)。

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## 2)Frobenius’ Theorem

假設奇異點 $x = x_0$ 為微分方程式(1)之規則奇異點，則至少存在一個解其型式如下：

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \tag{5}$$

其中 $r$ 為一(由指標方程式，indicial equation)待定之常數。級數(5)至少在一區間 $0 < x - x_0 < R$ 內為收斂。

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## 3)指標方程式( indicial equation)

假設奇異點 $x=0$ 為微分方程式(1)之規則奇異點，則將式(2)乘上 $x^2$ 可得

$$x^2y'' + x\left[xP(x)\right]y' + x^2Q(x)y = 0 \tag{6}$$

或

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{7}$$

其中，函數 $p(x)$, $q(x)$ 在 $x=0$ 皆為解析(可以一 power series 展開代表)，定義如下：

$$p(x) = xP(x) = \overline{a}_0 + \overline{a}_1x + \overline{a}_2x^2 + \ldots \tag{8}$$

$$q(x) = x^2Q(x) = \overline{b}_0 + \overline{b}_1x + \overline{b}_2x^2 + \ldots \tag{9}$$

將式(8)~(9)代入式(7)，可得

$$x^2y'' + x\left[\overline{a}_0 + \overline{a}_1x + \ldots\right]y' + \left[\overline{b}_0 + \overline{b}_1x + \ldots\right]y = 0 \tag{10}$$

另依 Frobenius' Theprem, 假設

$$y = \sum_{n=0}^{\infty} c_nx^{n+r} \tag{11}$$

$$\Rightarrow y' = \sum_{n=0}^{\infty} (n+r)c_nx^{n+r-1} \tag{12}$$

$$\Rightarrow y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_nx^{n+r-2} \tag{13}$$

將式(11)~(13)代入式(10)，可得

$$x^2y'' + x\left[\overline{a}_0 + \overline{a}_1x + \ldots\right]y' + \left[\overline{b}_0 + \overline{b}_1x + \ldots\right]y$$
$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r} + \left[\overline{a}_0 + \overline{a}_1x + \ldots\right]\sum_{n=0}^{\infty}(n+r)c_nx^{n+r} + \left[\overline{b}_0 + \overline{b}_1x + \ldots\right]\sum_{n=0}^{\infty} c_nx^{n+r} \tag{14}$$

其中 $x^r$ 之係數為 $\left[r(r-1) + \overline{a}_0r + \overline{b}_0\right]c_0$, 惟 $c_0 \neq 0$ (?)，所以可得**指標方程式**

$$r(r-1) + \overline{a}_0r + \overline{b}_0 = 0 \tag{14}$$

**4)指標方程式之根 $r_1, r_2, (r_1 > r_2)$及其對應之級數解**

**Case1:** $r_1, r_2$ 為相異，且差不為整數，則存在二個線性獨立之解 $y_1, y_2$ 如下

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} = x^{r_1}(c_0 + c_1 x + c_2 x^2 + \ldots) \tag{15}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} = x^{r_2}(b_0 + b_1 x + b_2 x^2 + \ldots) \tag{16}$$

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**Case2:** $r_1, r_2$ 為相等，$(r_1 = r_2)$，則存在二個線性獨立之解 $y_1, y_2$ 如下：

$$r_1 = r_2 \Rightarrow \ r_1 = r_2 = \frac{1-\bar{a}_0}{2} \quad \text{（由式(14)得知）} \tag{17}$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} = x^{r_1}(c_0 + c_1 x + c_2 x^2 + \ldots),\ c_0 \neq 0 \tag{18}$$

$$\begin{aligned} y_2(x) &= y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \\ &= y_1(x)\ln x + x^{r_2}(b_0 + b_1 x + b_2 x^2 + \ldots) \end{aligned} \tag{19}$$

**推導(reduction of order) (參見 p.A67~69, E. Kreyszig 8th Edition)**

$$\Rightarrow y_2(x) = u(x) y_1(x) \tag{20}$$

$$\Rightarrow y_2' = u' y_1 + u y_1' \tag{21}$$

$$\Rightarrow y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \tag{22}$$

$$\Rightarrow x^2 y'' + x p(x) y' + q(x) y = 0 \tag{23}$$

$$\Rightarrow x^2(u'' y_1 + 2u' y_1' + u y_1'') + x p(x)(u' y_1 + u y') + q(x) u y_1 = 0 \tag{24}$$

$$\Rightarrow u\left[x^2 y_1'' + x p(x) y' + q(x) y_1\right] + x^2 u'' y_1 + 2x^2 u' y_1' + x p(x) u' y_1 = 0 \tag{25}$$

$$\Rightarrow x^2 u'' y_1 + 2x^2 u' y_1' + x p(x) u' y_1 = 0 \tag{26}$$

因為 $p(x) = xP(x) = \bar{a}_0 + \bar{a}_1 x + \bar{a}_2 x^2 + \ldots$

$$\Rightarrow u'' + 2u' \frac{y_1'}{y_1} + \frac{p(x)u'}{x} = 0 \tag{27}$$

$$\Rightarrow u'' + \left( 2\frac{y_1'}{y_1} + \frac{\bar{a}_0}{x} + \bar{a}_1 + x\bar{a}_2 + \ldots \right) u' = 0 \tag{28}$$

所以

$$\begin{aligned} \frac{y_1'}{y_1} &= \frac{x^{r_1-1}\left[r_1c_0 + (r_1+1)c_1x + \ldots\right]}{x^{r_1}\left[c_0 + c_1x + c_2x^2 + \ldots\right]} \\ \Rightarrow \quad &= \frac{\left[r_1c_0 + (r_1+1)c_1x + \ldots\right]}{x\left[c_0 + c_1x + c_2x^2 + \ldots\right]} \\ &= \frac{r_1}{x} + \ldots \end{aligned} \tag{29}$$

若將式(22)代入式(21),可得

$$\begin{aligned} & u'' + \left( 2\frac{y_1'}{y_1} + \frac{\bar{a}_0}{x} + \bar{a}_1 + x\bar{a}_2 + \ldots \right) u' \\ & = u'' + \left( \frac{2r_1 + \bar{a}_0}{x} + \ldots \right) u' = 0 \end{aligned} \tag{30}$$

$$\Rightarrow u'' + \left( \frac{2r_1 + \bar{a}_0}{x} + \ldots \right) u' = 0 \tag{31}$$

而由式(17)，式(31)可改寫成

$$\Rightarrow \frac{u''}{u'} = -\frac{1}{x} + \ldots \tag{32}$$

積分式(32)，可得

$$\Rightarrow \ln u' = -\ln x + \ldots \tag{33}$$

再改寫式(33)為

$$\Rightarrow u' = \frac{1}{x} e^{(\ldots)} \tag{34}$$

再積分式(34)，可得

$$\Rightarrow u = \ln x + k_1x + k_2x^2 + \ldots \tag{35}$$

最後得到 $y_2$

$$\Rightarrow \begin{aligned} y_2(x) &= u y_1(x) \\ &= y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+r^2} \end{aligned} \quad (36)$$

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**Case3:**

**If** $r_1 - r_2 = N$ where $N$ is a positive integer, then there exist two linearly independent solutions of the form:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0 \quad (37)$$

$$y_2(x) = C y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad b_0 \neq 0 \quad (38)$$

**推導(reduction of order) (參見 p.A67~69, E. Kreyszig 8th Edition)**

$$\bar{a}_0 - 1 = -(r_1 + r_2) = -(r_1 + r_1 + N) = -2r_1 + N \text{（由式(14)得知）} \quad (39)$$

$$\Rightarrow 2r_1 + \bar{a}_0 = N + 1 \quad (40)$$

由式(40)，並接續式(31)

$$\Rightarrow \begin{aligned} \frac{u''}{u'} &= -\left(\frac{2r_1 + \bar{a}_0}{x} + \ldots\right) \\ &= -\left(\frac{N+1}{x} + \ldots\right) \end{aligned} \quad (41)$$

積分式(41)可得

$$\Rightarrow \ln u' = -(N+1)\ln x + \ldots \quad (42)$$

同樣再積分式(42)可得

$$\Rightarrow u' = x^{-(N+1)} e^{(\ldots)} \quad (43)$$

$$\Rightarrow u' = \frac{1}{x^{N+1}} + \frac{k_1}{x^N} + \ldots + \frac{k_{N-1}}{x^2} + \frac{k_N}{x} + k_{N+1} + k_{N+2}x + \ldots \quad (44)$$

$$\Rightarrow u = k_N \ln x + \left(-\frac{1}{N x^N} - \ldots - \frac{k_{N-1}}{x} + k_{N+1}x + \ldots\right) \quad (45)$$

所以可得到 $y_2(x) = C y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad b_0 \neq 0$