若 $r_1 - r_2 = N$ ，N 為正整數，則存在二個線性獨立之解 y_1，y_2 如下：

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0 \tag{1}$$

$$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad b_0 \neq 0 \tag{2}$$

注意：$r_1 > r_2$, n = **0** ～ ∞, C 也可能爲零，所以也可能有二個 **Frobenius solution**。

例題：$xy'' - y = 0$ （取自 example 4.15, O'Neil, 5th Edition)

$$y'' - \frac{1}{x} y = 0 \tag{3}$$

分母為零， $x = 0$ 為微分方程式(3)之奇異點(singular point)。

$$p(x) = xP(x) = 0 \tag{4}$$

$$q(x) = x^2 Q(x) = -x \tag{5}$$

函數 $p(x)$，$q(x)$ 現為多項式，所以在 $x = 0$ 皆為解析(analytic)。奇異點 $x = 0$ 為微分方程式(3)之規則奇異點(regular singular point)。

規則奇異點➔**Frobenius method**

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

➔ $$xy'' - y = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

➔ $$x^r \left[\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \right] = 0$$

k=n-1, k=n

➔ $$x^r \left\{ r(r-1) c_0 x^{-1} + \sum_{n=0}^{\infty} \left[(n+1+r)(n+r) c_{n+1} - c_n \right] x^n \right\} = 0$$

指標方程式(最低指標 x^{r-1} 的係數，且 $c_0 \neq 0$) ➔ $r(r-1) = 0$

➔ $r_1 = 1,\ r_2 = 0$ 二根之差爲正整數

➔ $(n+1+r)(n+r)c_{n+1} - c_n = 0$, $n = 0,\ 1,\ 2,\ 3, \ldots$

➔ $c_{n+1} = \dfrac{1}{(n+1+r)(n+r)} c_n$

當 $r_1 = 1$

➔ $c_{n+1} = \dfrac{1}{(n+1)(n+2)} c_n$

➔ $c_1 = \dfrac{1}{1(2)} c_0$

➔ $c_2 = -\dfrac{1}{4} c_1 = \dfrac{1}{2(3)} c_1 = \dfrac{1}{2(2)(3)} c_0$

➔ $c_3 = \dfrac{1}{3(4)} c_2 = \dfrac{1}{2(3)(2)(3)(4)} c_0$

➔ $c_n = \dfrac{1}{n!(n+1)!} c_0$, $n = 1,\ 2,\ 3, \ldots$

The **Frobenius solution** we have found is

$$\begin{aligned} y_1(x) &= c_0 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1} \\ &= c_0 \left[x + \frac{1}{2} x^2 + \frac{1}{12} x^3 + \frac{1}{144} x^4 + \ldots \right] \end{aligned} \tag{6}$$

當 $r_2 = 0$

➔ $y_2(x) = C y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^n$

➔ $x \left[C y_1'' \ln(x) + 2 C y_1' \dfrac{1}{x} - C y_1' \dfrac{1}{x^2} + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} \right] - C y_1 \ln(x) - \sum_{n=0}^{\infty} b_n x^n = 0$

$$\Rightarrow C\ln(x)\left[xy_1'' - y_1\right] + x\left[2Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=2}^{\infty} n(n-1)b_n x^{n-2}\right] - \sum_{n=0}^{\infty} b_n x^n = 0 \qquad (7)$$

其中 $y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1}$ 爲微分方程式第一個解，所以在式(7)中

$$C\ln(x)\left[xy_1'' - y_1\right] = 0$$

而簡單起見，可令式(7)中 $c_0 = 1$

$$\Rightarrow 2C\sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n - C\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^n + \sum_{n=2}^{\infty} b_n n(n-1)x^{n-1} - \sum_{n=0}^{\infty} b_n x^n = 0$$

$$\Rightarrow 2C\sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n - C\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^n + \sum_{n=1}^{\infty} b_{n+1}(n+1)nx^n - \sum_{n=0}^{\infty} b_n x^n = 0$$

$$\Rightarrow (2C - C - b_0)x^0 + \sum_{n=1}^{\infty}\left[\frac{2C}{(n!)^2} - \frac{C}{n!(n+1)!} + b_{n+1}(n+1)n - b_n\right]x^n = 0 \qquad (8)$$

同樣地，在式(8)中每個 x 的係數皆爲零

$$\Rightarrow C = b_0$$

$$\Rightarrow \frac{2C}{(n!)^2} - \frac{C}{n!(n+1)!} + n(n+1)b_{n+1} - b_{n-1} = 0,$$

$$\Rightarrow b_{n+1} = \frac{1}{n(n+1)}\left[b_n - \frac{(2n+1)C}{n!(n+1)!}\right], \quad n = 1,\ 2,\ 3,\ldots$$

所以第二個解爲 (簡單起見，先令 $b_0 = 1$ $\Rightarrow C = b_0 = 1$；並取 $b_1 = 0$ $\Rightarrow$特解)

$$\Rightarrow y_2(x) = y_1(x)\ln(x) + 1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 - \frac{35}{1728}x^4 - \ldots$$

注意這不是 **Frobenius solution.**

因此通解爲

$$
\begin{aligned}
y(x) &= C_1 y_1(x) + C_2 y_2(x) \\
&= \left[C_1 + C_2 \ln(x)\right] \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1} + C_2 \left[1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 - \frac{35}{1728}x^4 - \ldots\right]
\end{aligned}
$$