HOMEWORK #10s (Chapter 8 Exercises--- Matrices)

1) Use either Gaussian elimination or Gauss-Jordan elimination to solve the given system

$$\begin{aligned} x_1 + \qquad x_3 - x_4 &= 1 \\ 2x_2 + x_3 + x_4 &= 3 \\ x_1 - x_2 + \qquad x_4 &= -1 \\ x_1 + x_2 + x_3 + x_4 &= 2 \end{aligned}$$

(page 362, Problem 17)

ANS $\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right) \xrightarrow[\text{operations}]{\text{row}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

The solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$.

2) Find the rank of the given matrix

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & 2/3 & 3 & 1/3 \\ 6 & 6 & 6 & 12 & 0 \end{pmatrix}$$

(page 367, Problem 9)

ANS $\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & 2/3 & 3 & 1/3 \\ 6 & 6 & 6 & 12 & 0 \end{pmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{pmatrix} 1 & 1/2 & 1/3 & 3/2 & 1/6 \\ 0 & 1 & 4/3 & 1 & -1/3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; The rank is 3.

3) Evaluate the determinant of the given matrix by cofactor expansion

$$\begin{pmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

(page 373, Problem 19)

ANS $\begin{pmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 4\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0$

4) Find the inverse of the given matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$

(page 390, Problem 25)

ANS

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{operations}]{\text{row}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5/2 & 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & 1/3 & -1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1 & 1/2 & 1/2 \end{array}\right) \xrightarrow[\text{operations}]{\text{row}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/2 & -2/3 & -1/6 & 7/6 \\ 0 & 1 & 0 & 0 & 1 & 1/3 & 1/3 & -4/3 \\ 0 & 0 & 1 & 0 & 0 & -1/3 & -1/3 & 1/3 \\ 0 & 0 & 0 & 1 & -1/2 & 1 & 1/2 & 1/2 \end{array}\right); \quad A^{-1} = \begin{pmatrix} -1/2 & -2/3 & -1/6 & 7/6 \\ 1 & 1/3 & 1/3 & -4/3 \\ 0 & -1/3 & -1/3 & 1/3 \\ -1/2 & 1 & 1/2 & 1/2 \end{pmatrix}$$

5) Find the eigenvalues and eigenvectors of the given matrix

a) (page 400, Problem 19)

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

ANS We solve $\det(A-\lambda I) = \begin{pmatrix} -\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ 1 & 1 & -1-\lambda \end{pmatrix} = -(\lambda+1)(\lambda^2+1) = 0$

For $\lambda_1 = -1$ we have $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So that $k_1 = k_3$ and $k_2 = -k_3$. If $k_3 = 1$ then $K_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. For $\lambda_2 = i$ we have

$$\left(\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 1 & -1-i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So that $k_1 = ik_3$ and $k_2 = k_3$. If $k_3 = 1$ then $K_2 = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}$ and $K_3 = \bar{K}_2 = \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}$.

b) (page 400, Problem 21)

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{pmatrix}$$

ANS We solve $\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 5-\lambda & 6 \\ 0 & 0 & -7-\lambda \end{pmatrix} = -(\lambda - 1)(\lambda - 5)(\lambda + 7) = 0$

For $\lambda_1 = 1$ we have $\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So that $k_2 = k_3 = 0$. If $k_1 = 1$ then $K_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For $\lambda_2 = 5$ we have

$$\left(\begin{array}{ccc|c} -4 & 2 & 3 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -12 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So that $k_3 = 0$ and $k_2 = 2k_1$. If $k_3 = 1$ then $K_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. For $\lambda_3 = -7$ we have

$$\begin{pmatrix} 8 & 2 & 3 & | & 0 \\ 0 & 12 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/4 & | & 0 \\ 0 & 1 & 1/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

So that $k_1 = -1/4k_3$ and $k_2 = -1/2k_3$. If $k_3 = 4$ then $K_3 = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$.