HOMEWORK #6s (Chapter 3 Exercises--- Variation of Parameters, Cauchy-Euler Equation)

1) Solve the given differential equation by variation of parameters.

$y'' - y = \cosh(x)$ (page 136, Problem 7)

ANS ➔ $f(x) = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$m^2 - 1 = 0$ ➔ $m = \pm 1$

➔ $y_1 = e^x$, $y_2 = e^{-x}$, $W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^{x-x} - e^{x-x} = -2$

➔ $y_c = c_1 e^x + c_2 e^{-x}$

➔ $y_p = u_1 e^x + u_2 e^{-x}$

$$u_1' = -\frac{y_2 f(x)}{W} = -\frac{e^{-x}}{-2}\frac{e^x + e^{-x}}{2} = \frac{e^{-x+x} + e^{-x-x}}{4} = \frac{1 + e^{-2x}}{4}$$

➔ $u_1 = \dfrac{x}{4} - \dfrac{e^{-2x}}{8}$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^x}{-2}\frac{e^x + e^{-x}}{2} = -\frac{e^{2x} + 1}{4}$$

➔ $u_2 = -\dfrac{x}{4} - \dfrac{e^{2x}}{8}$

$$y_p = u_1 e^x + u_2 e^{-x} = \left(\frac{x}{4} - \frac{e^{-2x}}{8}\right)e^x + \left(-\frac{x}{4} - \frac{e^{2x}}{8}\right)e^{-x}$$

➔ $$= \frac{x}{2}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{1}{4}\left(\frac{e^{-x} + e^x}{2}\right)$$

$$= \frac{x}{2}\sinh(x) - \frac{1}{4}\left(\frac{e^{-x} + e^x}{2}\right)$$

$$y = y_c + y_p = c_1 e^x + c_2 e^{-x} + \frac{x}{2}\sinh(x) - \frac{1}{4}\left(\frac{e^x + e^{-x}}{2}\right)$$

➔ $$= \left(c_1 - \frac{1}{8}\right)e^x + \left(c_2 - \frac{1}{8}\right)e^{-x} + \frac{x}{2}\sinh(x)$$

$$= C_1 e^x + C_2 e^{-x} + \frac{x}{2}\sinh(x)$$

這與我 MAIL 給你們的 particular sol 檔案內要交待的觀念一樣，也就是，有時特解含有與補函數相同之項次時,常合併於補函數。

Solve the given differential equations.

2) $x^2y'' + 5xy' + 4y = 0$ (page 141, Problem 11)

ANS The auxiliary equation is $m^2 + 4m + 4 = (m+2)^2 = 0$ so that $y = c_1x^{-2} + c_2x^{-2}\ln x$.

3) $3x^2y'' + 6xy' + y = 0$ (page 141, Problem 13)

ANS The auxiliary equation is $3m^2 + 3m + 1 = 0$ so that

$y = x^{-1/2}[c_1\cos(\frac{\sqrt{3}}{6}\ln x) + c_2\sin(\frac{\sqrt{3}}{6}\ln x)]$.

4) Solve the given differential equation by variation of parameters.

$x^2y'' - xy' + y = 2x$ (page 141, Problem 21)

ANS The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$ so that $y_c = c_1x + c_2x\ln x$ and

$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$. Identifying $f(x) = 2/x$ we obtain $u_1' = -2\ln x/x$ and

$u_2' = 2/x$. Then $u_1 = -(\ln x)^2$, $u_2 = 2\ln x$, and

$y = c_1x + c_2x\ln x - x(\ln x)^2 + 2x(\ln x)^2 = c_1x + c_2x\ln x + x(\ln x)^2$.

5) Solve the given initial-value problem.

$xy'' + y' = x,\ y(1) = 1,\ y'(1) = -\frac{1}{2}$ (page 142, Problem 27)

ANS The auxiliary equation is $m^2 = 0$ so that $y_c = c_1 + c_2\ln x$ and

$W(1, \ln x) = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}$. Identifying $f(x) = 1$ we obtain $u_1' = -x\ln x$ and $u_2' = x$. Then

$u_1 = \frac{1}{4}x^2 - \frac{1}{2}x^2\ln x$, $u_2 = \frac{1}{2}x^2$, and

$y = c_1 + c_2\ln x + \frac{1}{4}x^2 - \frac{1}{2}x^2\ln x + \frac{1}{2}x^2\ln x = c_1 + c_2\ln x + \frac{1}{4}x^2$. The initial conditions imply

$c_1 + \frac{1}{4} = 1$ and $c_2 + \frac{1}{2} = -\frac{1}{2}$. Thus, $c_1 = \frac{3}{4}$, $c_2 = -1$, and $y = \frac{3}{4} - \ln x + \frac{1}{4}x^2$.

6) Solve the given initial-value problem on the interval $(-\infty,0)$.

$4x^2y'' + y = 0,\ \ y(-1) = 2,\ \ y'(-1) = 4$ (page 142, Problem 37)

ANS The differential equation and initial conditions become $4t^2\dfrac{d^2y}{dt^2}+y=0$; $y(t)\big|_{t=1}=2$, $y'(t)\big|_{t=1}=-4$. The auxiliary equation is $4m^2-4m+1=(2m-1)^2=0$, so that $y=c_1t^{1/2}+c_2t^{1/2}\ln t$ and $y'=\dfrac{1}{2}c_1t^{-1/2}+c_2(t^{-1/2}+\dfrac{1}{2}t^{-1/2}\ln t)$. The initial conditions imply $c_1=2$ and $1+c_2=-4$. Thus $y=2t^{1/2}-5t^{1/2}\ln t=2(-x)^{1/2}-5(-x)^{1/2}\ln(-x),\ x<0$.