HOMEWORK #7s (Chapter 5 Exercises--- Solutions about Ordinary Points)

1) Rewrite the given expression as a single power series

$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$ (page 246, Problem 9)

ANS $$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} = 2\cdot 1\cdot c_1 x^0 + \underbrace{\sum_{n=2}^{\infty} 2nc_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} 6c_n x^{n+1}}_{k=n+1}$$

$$= 2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6c_{k-1}x^k = 2c_1 + \sum_{k=1}^{\infty}[2(k+1)c_{k+1} + 6c_{k-1}]x^k$$

2) Verify by direct substitution that the given power series is a particular solution of the indicated differential equations.

$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ $(x+1)y'' + y' = 0$ (page 246, Problem 11)

$$y' = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)x^{n-2}$$

$$(x+1)y'' + y' = (x+1)\sum_{n=2}^{\infty} (-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty} (-1)^{n+1}x^{n-1}$$

$$= \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)x^{n-1} + \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty} (-1)^{n+1}x^{n-1}$$

ANS $$= -x^0 + x^0 + \underbrace{\sum_{n=2}^{\infty} (-1)^{n+1}(n-1)x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=3}^{\infty} (-1)^{n+1}(n-1)x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty} (-1)^{n+1}x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+2} kx^k + \sum_{k=1}^{\infty} (-1)^{k+3}(k+1)x^k + \sum_{k=1}^{\infty} (-1)^{k+2}x^k$$

$$= \sum_{k=1}^{\infty} [(-1)^{k+2}k - (-1)^{k+2}k - (-1)^{k+2} + (-1)^{k+2}]x^k = 0$$

3) Find two power series solutions of the given differential equation about the ordinary point $x = 0$

$y'' - 2xy' + y = 0$ (page 246, Problem 15)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - 2xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k\,=\,n\,-\,2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k\,=\,n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k\,=\,n}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - (2k-1)c_k]x^k = 0$$

Thus $2c_2 + c_0 = 0, \quad (k+2)(k+1)c_{k+2} - (2k-1)c_k = 0$

and $\quad c_2 = -\frac{1}{2}c_0, \quad c_{k+2} = \frac{2k-1}{(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \cdots$

$$k = 1 \to c_3 = \frac{1}{3\cdot 2}c_1$$

$$k = 2 \to c_4 = \frac{3}{4\cdot 3}c_2 = -\frac{1}{8}c_0$$

$$k = 3 \to c_5 = \frac{5}{5\cdot 4}c_3 = \frac{1}{24}c_1$$

$$k = 4 \to c_6 = \frac{7}{6\cdot 5}c_4 = -\frac{7}{336}c_0$$

$$k = 5 \to c_7 = \frac{9}{7\cdot 6}c_5 = \frac{1}{112}c_1$$

$$\therefore y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots.$$

$$= c_0[1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{336}x^6 + \cdots] + c_1[x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots]$$

$$= c_0 y_1 + c_1 y_2$$

4) Find two power series solutions of the given differential equation about the ordinary point $x = 0$

$(x-1)y'' + y' = 0$ (page 246, Problem 19)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x-1)y'' + y' = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty} (k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k \quad \text{Thus}$$

$$= -2c_2 + c_1 + \sum_{k=1}^{\infty} [(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1})]x^k$$

$$= 0$$

$$-2c_2 + c_1 = 0, \quad (k+1)^2 c_{k+1} - (k+2)(k+1)c_{k+2} = 0$$

and $c_2 = \frac{1}{2}c_1, \quad c_{k+2} = \frac{k+1}{k+2}c_{k+1}, \quad k = 1, 2, 3, \cdots$

$$k = 1 \rightarrow c_3 = \frac{2}{3}c_1$$

$$k = 2 \rightarrow c_4 = \frac{3}{4}c_3 = \frac{1}{4}c_1$$

$$k = 3 \rightarrow c_5 = \frac{4}{5}c_3 = \frac{1}{5}c_1$$

$$k = 4 \rightarrow c_6 = \frac{5}{6}c_4 = \frac{1}{6}c_1$$

$$\therefore y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$= c_0[1] + c_1[x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots]$$

$$= c_0 y_1 + c_1 y_2$$

5) Use the power series method to solve the given initial-value problem.

$(x-1)y'' - xy' + y = 0, \;\; y(0) = -2, \;\; y'(0) = 6$ (page 246, Problem 25)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x-1)y'' - xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=1}^{\infty} (k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{n=1}^{\infty} kc_k x^k + \sum_{n=0}^{\infty} c_k x^k$$

$$= -2c_2 + c_0 + \sum_{k=1}^{\infty} [(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} - (k-1)c_k]x^k$$

$$= 0$$

Thus $-2c_2 + c_0 = 0, \quad -(k+2)(k+1)c_{k+2} + (k-1)kc_{k+1} - (k-1)c_k = 0$

and $c_2 = \frac{1}{2}c_0, \quad c_{k+2} = \frac{kc_{k+1}}{k+2} - \frac{(k-1)c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \cdots$

$$k = 1 \to c_3 = \frac{1}{3}c_2 - \frac{0 \cdot c_1}{3 \cdot 2} = \frac{1}{6}c_0$$

$$k = 2 \to c_4 = \frac{2}{4}c_3 - \frac{1 \cdot c_2}{4 \cdot 3} = \frac{1}{24}c_0$$

$$k = 3 \to c_5 = \frac{3}{5}c_4 - \frac{2 \cdot c_3}{5 \cdot 4} = \frac{1}{120}c_0$$

$$\therefore y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$= c_0[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots] + c_1[x]$$

$$= c_0 y_1 + c_1 y_2$$

$$y' = c_0[x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots] + c_1$$

The initial conditions imply $c_0 = -2$ and $c_1 = 6$, so

$$y = -2(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots) + 6x = 8x - 2e^x$$

6) Use the power series method to solve the given initial-value problem.

$y'' - 2xy' + 8y = 0, \; y(0) = 3, \; y'(0) = 0$ (page 246, Problem 27)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - 2xy' + 8y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k\,=\,n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k\,=\,n} + 8\underbrace{\sum_{n=1}^{\infty} c_n x^n}_{k\,=\,n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + 8\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + 8c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (8-2k)c_k]x^k$$

$$= 0$$

Thus $2c_2 + 8c_0 = 0, \quad (k+2)(k+1)c_{k+2} + (8-2k)c_k = 0$

and $c_2 = -4c_0, \quad c_{k+2} = \dfrac{2k-8}{(k+2)(k+1)}c_k, \quad k = 1,2,3,\cdots$

$$k = 1 \rightarrow c_3 = \frac{-6}{3\cdot 2}c_1 = -c_1$$

$$k = 2 \rightarrow c_4 = \frac{-4}{4\cdot 3}c_2 = \frac{4}{3}c_0$$

$$k = 3 \rightarrow c_5 = \frac{-2}{5\cdot 4}c_3 = \frac{1}{10}c_1$$

$$k = 4 \rightarrow c_6 = \frac{0}{6\cdot 5}c_4 = 0$$

$$\therefore\ y = c_0[1 - 4x^2 + \frac{4}{3}x^4] + c_1[x - x^3 + \frac{1}{10}x^5 + \cdots]$$

$$y' = c_0[-8x + \frac{16}{3}x^3] + c_1[1 - 3x^2 + \frac{1}{2}x^4 + \cdots]$$

The initial conditions imply $c_0 = 3$ and $c_1 = 0$, so

$$y = 3(1 - 4x^2 + \frac{4}{3}x^4) = 3 - 12x^2 + 4x^4$$