HOMEWORK #8s (Chapter 5 Exercises--- Solutions about Singular Points)

1) Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$$x^3(x^2-25)(x-2)^2 y'' + 3x(x-2)y' + 7(x+5)y = 0$$ (page 254, Problem 9)

ANS Irregular singular point: $x=0$; regular singular points: $x=2, \pm 5$

2) Put the given differential equation into the form (3), *on page 247 of the textbook*, for each regular singular point of the equation. Identify the functions $p(x)$ and $q(x)$.

$$(x^2-1)y'' + 5(x+1)y' + (x^2-x)y = 0$$ (page 254, Problem 11)

ANS Writing the differential equation in the form $y'' + \frac{5}{x-1}y' + \frac{x}{x+1}y = 0$, we see that $x_0 = 1$ and $x_0 = -1$ are regular singular points. For $x_0 = 1$ the differential equation can be put in the form $(x-1)^2 y'' + 5(x-1)y' + \frac{x(x-1)^2}{x+1}y = 0$. In this case $p(x) = 5$ and $q(x) = x(x-1)^2/(x+1)$. For $x_0 = -1$ the differential equation can be put in the form $(x+1)^2 y'' + 5(x+1)\frac{x+1}{x-1}y' + x(x+1)y = 0$. In this case $p(x) = 5(x+1)/(x-1)$ and $q(x) = x(x+1)$.

3) In this problem, $x=0$ is a regular singular point of the given differential equation. Use the general form of the indicial equation in (14), *on page 251 of the textbook*, to find the indicial roots of the singularity. Without solving, discuss the number of series solutions you would expect to find using the method of Frobenius.

$$x^2 y'' + (\tfrac{5}{3}x + x^2)y' - \tfrac{1}{3}y = 0$$ (page 255, Problem 13)

ANS We identify $P(x) = 5/3x + 1$ and $Q(x) = -1/3x^2$, so that $p(x) = xP(x) = \frac{5}{3} + x$ and $q(x) = x^2 Q(x) = -\frac{1}{3}$. Then $a_0 = \frac{5}{3}$, $b_0 = -\frac{1}{3}$, and the indicial equation is $r(r-1) + \frac{5}{3}r - \frac{1}{3} = r^2 + \frac{2}{3}r - \frac{1}{3} = \frac{1}{3}(3r^2 + 2r - 1) = \frac{1}{3}(3r-1)(r+1) = 0$. The indicial roots are $\frac{1}{3}$ and -1. Since these do not differ by an integer we expect to find two series solutions using the method of Frobenius.

4) In this problem, $x=0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Using the method of Frobenius to obtain two linearly independent series solutions about $x=0$. Form the general solution on $(0,\ \infty)$. $9x^2y''+9x^2y'+2y=0$ (page 255, Problem 23)

ANS Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$9x^2y''+9x^2y'+2y=(9r^2-9r+2)c_0x^r+\sum_{k=1}^{\infty}[9(k+r)(k+r-1)c_k+2c_k+9(k+r-1)c_{k-1}]x^{k+r}=0$$

, which implies $9r^2-9r+2=(3r-1)(3r-2)=0$, and $[9(k+r)(k+r-1)+2]c_k+9(k+r-1)c_{k-1}=0$. The indicial roots are $r=1/3$ and $r=2/3$.

For $r=1/3$ the recurrence relation is $c_k=-\dfrac{(3k-2)c_{k-1}}{k(3k-1)}$, $k=1,2,3\cdots$, and $c_1=-\dfrac{1}{2}c_0$, $c_2=\dfrac{1}{5}c_0$, $c_3=-\dfrac{7}{120}c_0$. For $r=2/3$ the recurrence relation is $c_k=-\dfrac{(3k-1)c_{k-1}}{k(3k+1)}$, $k=1,2,3\cdots$, and $c_1=-\dfrac{1}{2}c_0$, $c_2=\dfrac{5}{28}c_0$, $c_3=-\dfrac{1}{21}c_0$, The general solution on $(0,\infty)$ is

$$y=C_1x^{1/3}(1-\frac{1}{2}x+\frac{1}{5}x^2-\frac{7}{120}x^3+\cdots)+C_2x^{2/3}(1-\frac{1}{2}x+\frac{5}{28}x^2-\frac{1}{21}x^3+\cdots).$$

In this **problem5), 6)**, $x=0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Using the method of Frobenius to obtain at least one series solutions about $x=0$. Form the general solution on $(0,\ \infty)$.

5) $xy''+2y'-xy=0$ (page 255, Problem 25)

ANS Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy''+2y'-xy=(r^2+r)c_0x^{r-1}+(r^2+3r+2)c_1x^r+\sum_{k=2}^{\infty}[(k+r)(k+r-1)c_k+2(k+r)c_k-c_{k-2}]x^{k+r-1}=0$$

, which implies $r^2+r=r(r+1)=0$, $(r^2+3r+2)c_1=0$, and $(k+r)(k+r+1)c_k-c_{k-2}=0$.

The indicial roots are $r_1=0$ and $r_2=-1$, so $c_1=0$. For $r_1=0$ the recurrence relation is $c_k=\dfrac{c_{k-2}}{k(k+1)}$, $k=2,3,4\cdots$, and $c_2=\dfrac{1}{3!}c_0$, $c_3=c_5=c_7=\cdots=0$, $c_4=\dfrac{1}{5!}c_0$, $c_{2n}=\dfrac{1}{(2n+1)!}c_0$. For $r_2=-1$ the recurrence relation is $c_k=\dfrac{c_{k-2}}{k(k-1)}$, $k=2,3,4\cdots$, and

$c_2 = \frac{1}{2!}c_0$, $c_3 = c_5 = c_7 = \cdots = 0$, $c_4 = \frac{1}{4!}c_0$, $c_{2n} = \frac{1}{(2n)!}c_0$, The general solution on $(0,\infty)$

is $y = C_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n} + C_2x^{-1}\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} = \frac{1}{x}[C_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1} + C_2\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}]$

$= \frac{1}{x}[C_1 \sinh x + C_2 \cosh x]$

6) $xy'' - xy' + y = 0$ (page 255, Problem 27)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$xy'' - xy' + y = (r^2 - r)c_0x^{r-1} + \sum_{k=0}^{\infty}[(k+r+1)(k+r)c_{k+1} - (k+r)c_k + c_k]x^{k+r} = 0$, which implies

$r^2 - r = r(r-1) = 0$, and $(k+r+1)(k+r)c_{k+1} - (k+r-1)c_k = 0$. The indicial roots are

$r_1 = 1$ and $r_2 = 0$. For $r_1 = 1$ the recurrence relation is $c_{k+1} = \frac{kc_k}{(k+2)(k+1)}$, $k = 0,1,2\cdots$,

and one solution is $y_1 = c_0 x$. A second solution is

$$y_2 = x\int \frac{e^{-\int -dx}}{x^2}dx = x\int \frac{e^x}{x^2}dx = x\int \frac{1}{x^2}(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots)dx$$

$$= x\int (\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots)dx = x[-\frac{1}{x} + \ln x + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{72}x^3 + \cdots]$$

$$= x\ln x - x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots$$

The general solution on $(0,\infty)$ is $y = C_1x + C_2y_2(x)$.

7) In this problem, $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the recurrence relation found by the method of Frobenius first with the largest root r_1. How many solutions did you find ? Next use the recurrence relation with the smaller root r_2. How many solutions did you find ? $xy'' + (x-6)y' - 3y = 0$ (page 255, Problem 31)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + (x-6)y' - 3y = (r^2 - 7r)c_0 x^{r-1} + \sum_{k=1}^{\infty}[(k+r)(k+r-1)c_k$$

$$+ (k+r-1)c_{k-1} - 6(k+r)c_k - 3c_{k-1}]x^{k+r-1} = 0$$

which implies $r^2 - 7r = r(r-7) = 0$, and $(k+r)(k+r-7)c_k + (k+r-4)c_{k-1} = 0$. The indicial roots are $r_1 = 7$ and $r_2 = 0$. For $r_1 = 7$ the recurrence relation is $(k+7)kc_k + (k+3)c_{k-1} = 0$, $k = 1, 2, 3\cdots$, or $c_k = -\dfrac{k+3}{k+7}c_{k-1}$, $k = 1, 2, 3\cdots$, Taking $c_0 \neq 0$ we obtain $c_1 = -\dfrac{1}{2}c_0$, $c_2 = \dfrac{5}{18}c_0$, $c_3 = -\dfrac{1}{6}c_0$, and so on. Thus, the indicial root $r_1 = 7$ yields a single solution . Now, for $r_2 = 0$ the recurrence relation is

$k(k-7)c_k + (k-4)c_{k-1} = 0$, $k = 1, 2, 3, \cdots$. Then $-6c_1 - 3c_0 = 0$, $-10c_2 - 2c_1 = 0$, $-12c_3 - c_2 = 0$, $-12c_4 + 0c_3 = 0 \Rightarrow c_4 = 0$, $-10c_5 + c_4 = 0 \Rightarrow c_5 = 0$, $-6c_6 + 2c_5 = 0 \Rightarrow c_6 = 0$, $0c_7 + 3c_6 = 0 \Rightarrow c_7$ is arbitrary, and $c_k = -\dfrac{k-4}{k(k-7)}c_{k-1}$, $k = 8, 9, 10\cdots$. Taking $c_0 \neq 0$ and $c_7 = 0$ we obtain $c_1 = -\dfrac{1}{2}c_0$, $c_2 = \dfrac{1}{10}c_0$, $c_3 = -\dfrac{1}{120}c_0$, $c_4 = c_5 = c_6 = \cdots = 0$. Taking $c_0 = 0$ and $c_7 \neq 0$ we obtain $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$, $c_8 = -\dfrac{1}{2}c_7$, $c_9 = \dfrac{5}{36}c_7$, $c_{10} = -\dfrac{1}{36}c_7$. In this case we obtain the two solutions

$$y_1 = 1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3 \quad \text{and} \quad y_2 = x^7 - \frac{1}{2}x^8 + \frac{5}{36}x^9 - \frac{1}{36}x^{10} + \cdots.$$