HOMEWORK #9s (Chapter 5 Exercises--- Two Special Equations)

1) Find the general solution of the given differential equation on $(0, \infty)$

$4x^2y'' + 4xy' + (4x^2 - 25)y = 0$ (page 264, Problem 3)

a) rewrite the given DE into the standard form of Bessel's equation

ANS $x^2y'' + xy' + [x^2 - (\frac{5}{2})^2]y = 0$

b) identify the value of υ

ANS $\upsilon = \frac{5}{2}, -\frac{5}{2}$

c) write out the Bessel function of the first kind of order υ and $-\upsilon$

ANS $J_{5/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\frac{7}{2}+n)}(\frac{x}{2})^{2n+\frac{5}{2}}$, $J_{-5/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\frac{-3}{2}+n)}(\frac{x}{2})^{2n-\frac{5}{2}}$

d) are your J_v and J_{-v} linearly independent ? why ?

ANS Linearly independent. Because r_1 and r_2 are distinct and do not differ by an integer.

e) write out the general solution of the given differential equation on $(0, \infty)$

ANS $y = c_1J_{5/2}(x) + c_2J_{-5/2}(x)$.

2) Find the general solution of the given differential equation on $(0, \infty)$

$xy'' + y' + xy = 0$ (page 264, Problem 5)

a) rewrite the given DE into the standard form of Bessel's equation

ANS $x^2y'' + xy' + (x^2 - 0^2)y = 0$

b) identify the value of υ

ANS $\upsilon = 0, 0$

c) write out the Bessel function of the first kind of order υ and $-\upsilon$

ANS $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)}(\frac{x}{2})^{2n}$, $J_{-0}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)}(\frac{x}{2})^{2n}$

d) are your J_v and J_{-v} linearly independent ? why ?

ANS Not linearly independent. When $\upsilon = m =$ integer, $J_{-m}(x) = (-1)^m J_m(x)$.

e) write out the Bessel function of the second kind of order υ

ANS $Y_0(x) = \lim_{\mu \to 0} Y_\mu(x)$

f) write out the general solution of the given differential equation on $(0,\ \infty)$

ANS $y = c_1 J_0(x) + c_2 Y_0(x)$

3) Find the general solution of the given differential equation on $(0,\ \infty)$

$x^2 y'' + xy' + (9x^2 - 4)y = 0$ (page 264, Problem 7)

a) write out the general solution of $x^2 y'' + xy' + (x^2 - 4)y = 0$

ANS $y = c_1 J_2(x) + c_2 Y_2(x)$

b) by referring to $x^2 y'' + xy' + (\lambda^2 x^2 - 4)y = 0$, identify the value of υ

ANS $\upsilon = 2, -2$

c) write out the general solution of $x^2 y'' + xy' + (9x^2 - 4)y = 0$ on $(0,\ \infty)$

ANS $y = c_1 J_2(3x) + c_2 Y_2(3x)$

4) Legendre's equation and Legendre polynomials (page 265, Problem 35)
a) write out the standard form of the Legendre's equation
ANS $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$
b) write out the general solution of the Legendre's equation
ANS

$$y_1(x) = c_0[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots]$$

$$y_2(x) = c_1[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots]$$

$$c_0 = (-1)^{n/2}\frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n},\quad c_1 = (-1)^{(n-1)/2}\frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}.$$

c) explain what the Legendre polynomials is
ANS The specific n th-degree polynomial solutions are called Legendre polynomials.

d) write out the Legendre polynomials $P_5(x)$, $P_6(x)$

ANS $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$, $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

e) write the differential equation for which $P_5(x)$ is a particular solution

ANS $(1 - x^2)y'' - 2xy' + 30y = 0$

f) write the differential equation for which $P_6(x)$ is a particular solution

ANS $(1 - x^2)y'' - 2xy' + 42y = 0$