Homogeneous Linear Equations with Constant Coefficients

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(常係數之齊次線性方程式)

我們將探討常係數之齊次線性微分方程式的通解之求法，首先其標準式如下：

Standard Form

A linear homogeneous ordinary differential equation with constant coefficients has the general form of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots a_1 y' + a_0 y = 0 \qquad (1)$$

where $a_n, a_{n-1}, \ldots a_1, a_0$ are all *constants*.

上式為常係數之齊次線性 n 階微分方程式的標準式，其係數 $a_n, a_{n-1}, \ldots a_1, a_0$ 皆為常數。

Homogeneous Linear 2nd Order ODE with Constant Coefficients

為簡化起見，先從常係數之齊次線性二階微分方程式談起，

A homogeneous linear second order ordinary differential equation with *constant coefficients* can be expressed as

$$ay'' + by' + cy = 0 \qquad (2)$$

由方程式型式觀察得知，其中因變數 y 之導函數與函數 y 具有相同型式，隱含因變數 y 為指數函數 e^{mx}． 將 $y = e^{mx}$ 代入微分方程式，可得以下之輔助方程式(或稱特徵方程式)。

Substitute the exponential function e^{mx} into the above differential equation, the **auxiliary equation** or **characteristic equation** of this differential equation is obtained

$$am^2 + bm + c = 0 \qquad (3)$$

因此，微分方程式現轉換成式(3)之代數方程式，而滿足這代數方程式之 m_1 及 m_2，可表示如式(4)：

This characteristic equation has two roots m_1 and m_2.

$$m_1, m_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (4)$$

由式(4)之結果，m_1及 m_2的值決定於式(3)之代數方程式的係數a,b,c，討論如下：

	Solutions of auxiliary equation or characteristic equation m_1，m_2	General Solution
1	m_1，$m_2 \in R$，$m_1 \neq m_2$ 二相異實根	$y(x)=c_1e^{m_1x}+c_2e^{m_2x}$ 通解型式
2	m_1，$m_2 \in R$，$m_1 = m_2 = m$ 二個實數重根	$y(x)=c_1e^{mx}+c_2xe^{mx}$ 注意通解之型式
3	$m_1=\alpha+i\beta$；$m_1=\alpha-i\beta$ where $\alpha,\ \beta\in R,\ \beta>0$ 共軛複數根	$y(x)=c_1e^{m_1x}+c_2e^{m_2x}$ $=e^{\alpha x}\left(C_1\cos(\beta x)+C_2\sin(\beta x)\right)$ 注意上式為複數型式通解及複數常係數 而下式為實數型式通解及實數常係數

本表修改自：**efunda**

當推廣至常係數之齊次線性 n 階微分方程式時，同樣可假設 $y=e^{mx}$，並代入微分方程式，得以下之 **n** 次之輔助方程式(或稱特徵方程式)。

<table>
<tr><th colspan="3">nth Order Linear Homogeneous ODE with Constant Coefficients:
$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... a_1 y' + a_0 y = 0$</th></tr>
<tr><td colspan="3">Auxiliary equation or Characteristic Equation: $a_n m^n + a_{n-1} m^{n-1} + ... a_1 m + a_0 = 0$</td></tr>
<tr><th colspan="2">Solutions of Auxiliary equation or Characteristic Equation
$m_1, m_2, m_3, ..., m_n$</th><th>General Solution</th></tr>
<tr><td>1</td><td>$m_1, m_2, m_3, ..., m_n$ are all different real numbers.
相異實根</td><td>$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$
通解型式(與二階相似)</td></tr>
<tr><td>2</td><td>$m_1, m_2, m_3, ..., m_k$ are k repeated real roots; others are different real numbers.
K 個實數重根，及(n-k)個相異實根</td><td>$y(x) = e^{m_1 x}\left(c_1 + c_2 x + c_3 x^2 + ... + c_k x^{k-1}\right) +$
$c_{k+1} e^{m_{k+1} x} + ... + c_n e^{m_n x}$
注意對應於 K 個實數重根之通解型式</td></tr>
<tr><td>3</td><td>$m_1, m_2, m_3, ..., m_k$ are k/2 pairs of complex conjugate roots $\alpha \pm i\beta$; others are different real numbers.
k 個共軛複數根，及(n-k) 個相異實根</td><td>$y(x) = e^{\alpha x}\left(C_1 + C_2 x + C_3 x^2 + ... + C_{\frac{k}{2}} x^{\frac{k}{2}-1}\right)\cos(\beta x) +$
$\left(C_1 + C_2 x + C_3 x^2 + ... + C_{\frac{k}{2}} x^{\frac{k}{2}-1}\right)\sin(\beta x) +$
$C_{k+1} e^{m_{k+1} x} + ... + C_n e^{m_n x}$
實數型式通解及實數常係數</td></tr>
</table>

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你須對照此表，應用於課本例題 3(三階)及例題 4(四階)(page 120)