Recall:

Linear: A differential equation is called linear if there are **no multiplications** among **dependent variables** and their **derivatives**. In other words, all coefficients are functions of independent variables.

Definition: Linear Equation

A first-order differential equation of the form

$$a_1(x)\ \frac{dy}{dx} + a_0(x)\ \ y = g(x)$$

is said to be a linear equation.

When g(x) = 0 the equation is called **homogeneous**, when otherwise the the DE is called **nonhomogeneous**.

The standard form of a linear first-order DE is

$$a_1(x)\ \frac{dy}{dx} + a_0(x)\ \ y = g(x) \Rightarrow \frac{dy}{dx} + p(x)\ \ y = f(x) \quad \text{-------(*)}$$

We seek a solution of equation (*) on an interval I for which both function $p(x)$ and $f(x)$ are continuous.

We attempt to find an **integrating factor** $\mu(x)$ such that

$$\mu\ \left(y' + p(x)\ \ y\right) = \mu\ \ f(x)$$

$\Rightarrow$ $(\mu\ y)' = \mu\ \ f(x)$ <== $(\mu\ y)' = \mu y' + \mu' y = \mu y' + \mu p(x) y$

$$\mu' = \mu p(x)$$

$$\frac{d\mu}{\mu} = p(x)dx$$

$$\ln|\mu| = \int p(x)dx + C_1$$

$$|\mu| = e^{\int p(x)dx + C_1}$$

$$\mu = \pm e^{C_1} e^{\int p(x)dx} = C_2 e^{\int p(x)dx}$$

$\Rightarrow$ $$\frac{d(\mu\ y)}{dx} = \mu\ \ f(x)$$

$\Rightarrow$ $$\int d(\mu\ y) = \int \mu\ \ f(x)dx + C_3$$

➔ $\mu\ y = \int \mu\ f(x)dx + C_3$

➔ $y = \mu^{-1} \int \mu\ f(x)dx + C_3\mu^{-1}$

➔ $y = \frac{1}{C_2} e^{\int p(x)dx} \int C_2 e^{\int p(x)dx}\ f(x)dx + C_3 \frac{1}{C_2} e^{-\int p(x)dx}$

➔ $y = e^{\int p(x)dx} \int e^{\int p(x)dx}\ f(x)dx + Ce^{-\int p(x)dx}$

Thus, we obtain the general solution of this DE,

$$\frac{dy}{dx} + p(x)\ y = f(x)$$

y(x) =y_h(x) + y_p(x) with

y_h(x) corresponding to the homogeneous version of the standard form (i.e. f(x) = 0), and with y_p(x) being a particular solution of the nonhomogeneous form (i.e. f(x) $\neq$ 0) of the DE.

Solving a linear first-order equation:

1) Recognize a linear first-order equation

$$a_1(x)\ \frac{dy}{dx} + a_0(x)\ y = g(x)$$

2) Reform it in the standard form

$$\frac{dy}{dx} + p(x)\ y = f(x)$$

3) Find an integrating factor

$$\mu = e^{\int p(x)dx}$$

4) Rewrite the linear equation as

$$\frac{d(\mu\ y)}{dx} = \mu\ f(x)$$

Or $$y = e^{\int p(x)dx} \int e^{\int p(x)dx}\ f(x)dx + Ce^{-\int p(x)dx}$$

Bernoulli's differential equation:

$$\frac{dy}{dx} + p(x)\ \ y = f(x)\ \ y^n$$

where n is any real number.

For $n = 0$

➔ $$\frac{dy}{dx} + p(x)\ \ y = f(x)$$ ➔ linear equation

For $n = 1$

➔ $$\frac{dy}{dx} + p(x)\ \ y = f(x)\ \ y$$ ➔ linear equation (or separable equation)

For $n \neq 0$ and $n \neq 1$

Let $u = y^{1-n}$

➔ $\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

➔ $$(1-n)y^{-n}\left(\frac{dy}{dx} + p(x)\ \ y\right) = (1-n)y^{-n}\left(f(x)\ \ y^n\right)$$

➔ $$(1-n)y^{-n}\frac{dy}{dx} + p(x)(1-n)y^{1-n} = (1-n)f(x)$$

➔ $$\frac{du}{dx} + p(x)(1-n)u = (1-n)f(x)$$ **linear in *u* and *x*.**

The Riccati Equation:

$$\frac{dy}{dx} = P(x) + Q(x)\ \ y + R(x)\ \ y^2$$

For $P(x) = 0$ ➔ Bernoulli's Equation with $n = 2$ ➔ $u = y^{1-2}$ ==> $y = \frac{1}{u}$

For $P(x) \neq 0$

➔Let $y = y_p + \frac{1}{u}$ with $y_p(x)$ a given particular solution.

$$\Rightarrow \frac{dy}{dx} = \frac{dy_p}{dx} - \frac{1}{u^2}\frac{du}{dx}$$

$$\Rightarrow \frac{dy_p}{dx} - \frac{1}{u^2}\frac{du}{dx} = P(x) + Q(x)\left[y_p + \frac{1}{u}\right] + R(x)\left[y_p + \frac{1}{u}\right]^2$$

$$\Rightarrow \frac{dy_p}{dx} - \frac{1}{u^2}\frac{du}{dx} = P(x) + Q(x)\ y_p + Q(x)\ \frac{1}{u} + R(x)\ y_p{}^2 + 2R(x)\ y_p\frac{1}{u} + R(x)\ \frac{1}{u^2}$$

$$\Rightarrow -\frac{1}{u^2}\frac{du}{dx} = \left[Q(x)\ y_p + R(x)\ y_p{}^2 + P(x) - \frac{dy_p}{dx}\right] + \left(Q(x) + 2R(x)\ y_p\right)\frac{1}{u} + R(x)\ \frac{1}{u^2}$$

Since $y_p(x)$ is the particular solution $\Rightarrow$ $\frac{dy_p}{dx} = P(x) + Q(x)\ y_p + R(x)\ y_p{}^2$

$$\Rightarrow -\frac{1}{u^2}\frac{du}{dx} = \left(Q(x) + 2R(x)\ y_p\right)\frac{1}{u} + R(x)\ \frac{1}{u^2}$$

$$\Rightarrow \frac{du}{dx} = -\left(Q(x) + 2R(x)\ y_p\right)u - R(x)$$

$$\Rightarrow \frac{du}{dx} + \left(Q(x) + 2R(x)\ y_p\right)u = -R(x)$$ **linear in *u* and *x*.**

Linear Differential Equations

Differential Equation	General Solution / Simplifying Method
Linear differential equation: $\frac{dy}{dx}+p(x)\ y=f(x)$	$y=e^{\int p(x)dx}\int e^{\int p(x)dx}\ f(x)dx+Ce^{-\int p(x)dx}$ where $e^{\int p(x)dx}$ is the integrating factor.
Bernoulli's differential equation: $\frac{dy}{dx}+p(x)\ y=f(x)\ y^n$	Let $u=y^{1-n}$ ➔ $\frac{du}{dx}+p(x)(1-n)u=(1-n)f(x)$ which is a linear differential equation.
Ricatti's differential equation: $\frac{dy}{dx}=P(x)+Q(x)\ y+R(x)\ y^2$ which is the Bernoulli's differential equation with n=2 and a non-homogeneous term $P(x)$.	Let $y(x)=\frac{1}{u}+y_p(x)$ where $y_p(x)$ is the particular solution. ➔ $\frac{du}{dx}+\left(Q(x)+2R(x)\ y_p\right)u=-R(x)$

The table is *revised* from **eFunda**

EX:

1) $\frac{dy}{dx}+y=\sin(x)$ Ans: $y(x)=\frac{1}{2}\left[\sin(x)-\cos(x)\right]+ce^{-x}$

2) $\frac{dy}{dx}+\frac{1}{x}y=3x^2y^3$ Ans: $y(x)=\frac{1}{\sqrt{cx^2-6x^3}}$

3) $\frac{dy}{dx}=-\frac{1}{x}y^2+\frac{2}{x}y$ Ans: $y(x)=2+\frac{2}{cx^2-1}$