**Linear Ordinary Differential Equations**: A nth order linear ordinary differential equations have the general form of

N 階線性(對變數 $y$ 而言)常微分方程式通式可表示成以下型式：

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \qquad (1)$$

where $a_n(x), a_{n-1}(x),\ldots a_1(x), a_0(x)$ are all functions of $x$.

其中係數 $a_n(x), a_{n-1}(x),\ldots a_1(x), a_0(x)$ 是 $x$ 的函數。

This differential equation is *homogeneous* if $g(x)=0$. Otherwise, it is a *non-homogeneous* differential equation.

上述 N 階線性(對變數 $y$ 而言)常微分方程式，若 $g(x)=0$ 則稱為齊次(或均質)，否則 $g(x)\neq 0$ 稱為非齊次(或非均質) 微分方程式。

----------------------------------------------------------------------------------------------------------

## Linear Dependence and Independence

線性相依與線性獨立

**Linear Dependence**: Consider a set of functions $f_1(x), f_2(x),\ldots, f_n(x)$ is said to be linearly dependant on an interval $I$ if there exist constants $c_1, c_2,\ldots, c_n$ not all zero, such that $c_1 f_1(x)+c_2 f_2(x)+\ldots+c_n f_n(x)=0$ for every $x$ in the interval.

**Linear Independence**: Consider a set of functions $f_1(x), f_2(x),\ldots, f_n(x)$ defined on an interval $I$. If the only way to make the linear combination of these functions be zero, $c_1 f_1(x)+c_2 f_2(x)+\ldots+c_n f_n(x)=0$, is that all constants are zero, $c_1=0=c_2=\ldots=c_n=0$, this set of functions is called linearly independent on an interval $I$.

因此，我們判斷一組函數是線性獨立時，是當這組函數的線性組合(linear combination)為零時，其線性組合的係數須全為零；而假如其線性組合的係數不全為零時，仍可得這組函數的線性組合(linear combination)為零的情況，則這一組函數是線性相依。

**Solutions and Superposition**

解及線性疊加

**1)**本章在準備求解 n 階線性微分方程式時，”**策略**”是先針對齊次型式，藉由 **n 階線性齊次微分方程式** n 個解(線性獨立)的**線性組合**，得到此一 n 階線性齊次微分方程式的通解。

**Linear Combination of Solutions**: Consider a nth order linear homogeneous ordinary differential equations

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...a_1(x)\frac{dy}{dx}+a_0(x)y=0 \qquad (2)$$

If $y_1(x), y_2(x),..., y_n(x)$ are solutions of this **linear homogeneous** differential equation, their *linear combinations* are also solutions of this equation, i.e.,

$$a_n(x)Y^{(n)}+a_{n-1}(x)Y^{(n-)}+...a_1(x)Y^{'}+a_0(x)Y=0 \qquad (3)$$

where $Y(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$.

2)考慮式(2) **n 階線性齊次微分方程式**，假如你得到其 n 個解 $y_1(x), y_2(x),..., y_n(x)$ ，則此 n 個解的線性組合 $Y(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$ 亦為該方程式的解。

-----------------------------------------------------------------------------------------------------------

**General Solutions of Linear Homogeneous Differential Equations**:
Consider a nth order linear homogeneous ordinary differential equations

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-)}+...a_1(x)y^{'}+a_0(x)y=0 \qquad (4)$$

If $y_1(x), y_2(x),..., y_n(x)$ are n *independent* solutions of this differential equation, their linear combinations form the **general solution** of this equation, i.e.,

$$y(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x) \qquad (5)$$

where $c_1, c_2,..., c_n$ are arbitrary constants.

3)所以求解 **n 階線性齊次微分方程式**的解時，你先得到其 n 個線性獨立的解 $y_1(x), y_2(x),..., y_n(x)$ ，則此 n 個解的線性組合 $y(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$ 為該方程式的通解，其中 $c_1, c_2,..., c_n$ 為任意常數。

———————————————————————————————————————

**Particular Solutions**: Consider a nth order linear non-homogeneous ordinary differential equations

$$a_n(x)y_p^{(n)} + a_{n-1}(x)y_p^{(n-)} + \ldots a_1(x)y_p' + a_0(x)y_p = g(x) \tag{6}$$

where $g(x) \neq 0$.

If $y_p(x)$ contains no arbitrary constants and satisfies this differential equation, $y_p(x)$ is called the **particular solution** of this equation.

4)接著，本章在準備求解 n 階線性微分方程式時，”**策略**”是先針對**非齊次**型式，求 n **階線性非齊次微分方程式的特解**(---滿足微分方程式，且不含常數)。

---

5)最後來到如何求得 **n 階線性非齊次微分方程式的通解**

6)首先，n **階線性非齊次微分方程式的特解** $y_p(x)$；以及所對應之 n 階線性齊次微分方程式的通解(補函數或補解) $y_c(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$，則 **n 階線性非齊次微分方程式的通解為** $y_p(x)$ 與 $y_c(x)$ 的疊加，亦即 $y(x) = y_c(x) + y_p(x)$。

**General Solutions of Linear Non-homogeneous Differential Equations**:

Consider a nth order linear non-homogeneous ordinary differential equations

$$a_n(x)y_p^{(n)} + a_{n-1}(x)y_p^{(n-)} + \ldots a_1(x)y_p' + a_0(x)y_p = g(x) \tag{7}$$

where $g(x) \neq 0$.

If $y_p(x)$ is the particular solution

$$a_n(x)y_p^{(n)} + a_{n-1}(x)y_p^{(n-)} + \ldots a_1(x)y_p' + a_0(x)y_p = g(x) \tag{8}$$

and $y_c(x)$, the complementary solution, is the general solution of the associated homogeneous differential equation

$$a_n(x)y_c^{(n)} + a_{n-1}(x)y_c^{(n-)} + \ldots a_1(x)y_c' + a_0(x)y_c = 0 \tag{9}$$

then the **general solution** of the **linear** non-homogeneous equation is the *superposition* of both particular and complementary solutions

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x) \end{aligned} \tag{10}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, $y_1(x), y_2(x), \ldots, y_n(x)$ are n independent solutions of the associated homogeneous equation.