Reduction of Order (降階法)

Consider a homogeneous linear second-order differential equation

$y'' + P(x)y' + Q(x)y = 0$ (1)

$P(x),\ Q(x)$ are continuous on some interval I

Suppose y_1 is a known solution on I and that $y_1 \neq 0$ for every x in the interval. We aim to seek a second solution y_2 so that y_1 and y_2 are linearly independent on the interval I.

第三章針對高階微分方程式之求解

基於解析方法的可行性，考慮齊次線性二階微分方程式的簡易案例。須注意的是，係數 $P(x),\ Q(x)$ 在定義區間內連續的敘述，在於支持後續(初始值問題)求解時之解存在及唯一的性質。

假如已知一個解 y_1 (暫且不管如何得到)，則依定理，可知微分方程式(1)存在二個線性獨立之解(爲什麼?)，則如何求得另一個解 y_2

Recall that if y_1 and y_2 are linearly independent, then the ratio $y_2(x)/y_1(x)$ is not a constant on the interval I.

由函數線性組合的線性相依、線性獨立之性質，可知若 y_1 and y_2 (線性組合)爲線性獨立，則 $y_2(x)/y_1(x)$ 不爲常數，則可設這 x 的函數的比值爲 $y_2(x)/y_1(x) = u(x)$。$u(x)$ 推導如下：

➔ $y_2(x)/y_1(x) = u(x)$, $y_2(x) = u(x)y_1(x)$

➔ $y_2' = uy_1' + y_1u'$

➔ $y_2'' = uy_1'' + 2y_1'u' + y_1u''$

➔ $y'' + P(x)y' + Q(x)y = uy_1'' + 2y_1'u' + y_1u'' + P(x)(uy_1' + y_1u') + Q(x)(uy_1)$

$$= u\left[y_1'' + P(x)y_1' + Q(x)y_1\right] + y_1u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

➔ $y_1u'' + \left(2y_1' + P(x)y_1\right)u' = 0$

Let $w = u'$

➔ $y_1w' + \left(2y_1' + P(x)y_1\right)w = 0$ a linear and separable DE (所以降階法將二階微分方程

式轉換爲一階微分方程式，降階也者!)

➔ $\frac{dw}{w}+2\frac{y_1'}{y_1}dx+P(x)dx=0$

➔ $\frac{dw}{w}+2\frac{dy_1}{y_1}+P(x)dx=0$

➔ $\ln|w|+2\ln|y_1|=-\int P(x)dx+C$

➔ $\ln|wy_1^2|=-\int P(x)dx+C$

➔ $wy_1^2=\pm e^{-\int P(x)dx+C}=\pm e^C e^{-\int P(x)dx}$

➔ $wy_1^2=C_1 e^{-\int P(x)dx}$

➔ $w=C_1\frac{e^{-\int P(x)dx}}{y_1^2}$

➔ $\frac{du}{dx}=w=C_1\frac{e^{-\int P(x)dx}}{y_1^2}$

➔ $\int du=C_1\int\frac{e^{-\int P(x)dx}}{y_1^2}dx+C_2$

➔ $u=C_1\int\frac{e^{-\int P(x)dx}}{y_1^2}dx+C_2$

For $C_1=1,\ C_2=0$ we get the simplest, but still acceptable, u

尋求最簡單之$u(x)$以簡化求得另一個解y_2之計算問題

➔ $u=\int\frac{e^{-\int P(x)dx}}{y_1^2}dx$

➔ $y_2=uy_1=y_1\int\frac{e^{-\int P(x)dx}}{y_1^2}dx$

➔the general solution $y=c_1y_1+c_2y_2$

這時相對於齊次線性二階微分方程式(1)，我們基於一已知解y_1，利用降階法求得另一個

解 y_2，y_1 and y_2 (線性組合)為線性獨立，構成線性二階微分方程式(1)解的基本組，也就是說，**齊次**線性二階微分方程式(1)的通解為 y_1 and y_2 的線性組合 $y = c_1 y_1 + c_2 y_2$。