Review Problems for Mid-term Exam IIIs

1) Find the radius of convergence and interval of convergence for the given power series.

(1) $\sum_{n=1}^{\infty} \dfrac{2^n}{n} x^n$ (problem 1, page 245)

ANS $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n / n}\right| = \lim_{n\to\infty}\dfrac{2n}{n+1}|x| = 2|x|$

The series is absolutely convergent for $2|x|<1$ *or* $|x|<1/2$. At $x=-1/2$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ converges by the alternating series test. At $x=1/2$, the series $\sum_{n=1}^{\infty}\dfrac{1}{n}$ is the harmonic series which diverges. Thus, the given series converges on $[-1/2,1/2)$.

(2) $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{10^k}(x-5)^k$ (problem 3, page 245)

ANS $\lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{(x-5)^{k+1}/10^{k+1}}{(x-5)^k/10^k}\right| = \lim_{k\to\infty}\dfrac{1}{10}|x-5| = \dfrac{1}{10}|x-5|$

The series is absolutely convergence for $\dfrac{1}{10}|x-5|<1,\quad |x-5|<10$, or on (-5,15). At $x=-5$, the series $\sum_{k=1}^{\infty}\dfrac{(-1)^k(-10)^k}{10^k}=\sum_{k=1}^{\infty}1$ diverges by the *k*-th term test. At $x=15$, the series $\sum_{k=1}^{\infty}\dfrac{(-1)^k 10^k}{10^k}=\sum_{k=1}^{\infty}(-1)^k$ diverges by the *k*-th term test. Thus, the series converges on (-5 , 15).

2) Verify by direct substitution that the given power series is a particular solution of the indicated differential equation

$$y=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n, \qquad (x+1)y'' + y' = 0$$ (page 246, Problem 11)

ANS $y' = \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}, \quad y'' = \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}$

$$(x+1)y'' + y' = (x+1)\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$= \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1} + \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$= -x^0 + x^0 + \underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=3}^{\infty}(-1)^{n+1}(n-1)x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty}(-1)^{k+2}kx^k + \sum_{k=1}^{\infty}(-1)^{k+3}(k+1)x^k + \sum_{k=1}^{\infty}(-1)^{k+2}x^k$$

$$= \sum_{k=1}^{\infty}[(-1)^{k+2}k - (-1)^{k+2}k - (-1)^{k+2} + (-1)^{k+2}]x^k$$

$$= 0$$

3) Find two power series solutions of the given differential equation about the ordinary point $x = 0$

$(x-1)y'' + y' = 0$ (page 246, Problem 19)

ANS $$(x-1)y'' + y' = \underbrace{\sum_{n=2}^{\infty}n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty}nc_n x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$$

$$= -2c_2 + c_1 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}]x^k = 0.$$

Thus

$$-2c_2 + c_1 = 0$$

$$(k+1)^2 c_{k+1} - (k+2)(k+1)c_{k+2} = 0$$

and

$$c_2 = \frac{1}{2}c_1$$

$$c_{k+2} = \frac{k+1}{k+2}c_{k+1}, \quad k = 1, 2, 3, \ldots$$

Choosing $c_0 = 1 \; and \; c_1 = 0$ we find $c_2 = c_3 = c_4 = \ldots = 0.$ For $c_0 = 0 \; and \; c_1 = 1$ we obtain

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{3}, \quad c_4 = \frac{1}{4},$$

and so on. Thus, two solutions are

$$y_1 = 1 \quad and \quad y_2 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots.$$

4) Use the power series method to solve the given initial-value problem.

$$(x-1)y'' - xy' + y = 0, \quad y(0) = -2, \quad y'(0) = 6 \qquad \text{(page 246, Problem 25)}$$

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x-1)y'' - xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=1}^{\infty} (k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= -2c_2 + c_0 + \sum_{k=1}^{\infty} [-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k]x^k = 0.$$

Thus

$$-2c_2 + c_0 = 0$$
$$-(k+2)(k+1)c_{k+2} + (k-1)kc_{k+1} - (k-1)c_k = 0$$

and

$$c_2 = \frac{1}{2}c_0$$

$$c_{k+2} = \frac{kc_{k+1}}{k+2} - \frac{(k-1)c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \cdots$$

Choosing $c_0 = 1 \; and \; c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{6}, \quad c_4 = 0$$

and so on. For $c_0 = 0 \; and \; c_1 = 1$ we obtain $c_2 = c_3 = c_4 = \cdots = 0.$ Thus,

$$y = C_1\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) + C_2 x$$

and

$$y' = C_1\left(x + \frac{1}{2}x^2 + \cdots\right) + C_2.$$

The initial conditions imply $C_1 = -2 \; and \; C_2 = 6,$ so

$$y = -2(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots) + 6x = 8x - 2e^x.$$

5) In this problem, $x = 0$ is a regular singular point of the given differential equation. Use the general form of the indicial equation in (14), *on page 251 of the textbook*, to find the indicial roots of the singularity. Without solving, discuss the number of series solutions you would expect to find using the method of Frobenius.

$x^2 y'' + (\frac{5}{3}x + x^2)y' - \frac{1}{3}y = 0$ (page 255, Problem 13)

ANS We identify $P(x) = 5/3x + 1 \; and \; Q(x) = -1/3x^2$, so that

$$p(x) = xP(x) = \frac{5}{3} + x \; and \; q(x) = x^2 Q(x) = -\frac{1}{3}.$$

Then $a_0 = \frac{5}{3}$, $b_0 = \frac{-1}{3}$, and the indicial equation is

$$r(r-1) + \frac{5}{3}r - \frac{1}{3} = r^2 + \frac{2}{3}r - \frac{1}{3} = \frac{1}{3}(3r^2 + 2r - 1) = \frac{1}{3}(3r-1)(r+1) = 0.$$

The indicial roots are $\frac{1}{3} \; and \; -1$. Since these do not differ by an integer we expect to find two series solutions using the method of Frobenius.

In the **problem 6), 7)** $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Using the method of Frobenius to obtain at least one series solutions about $x = 0$. Form the general solution on $(0, \infty)$.

6) $xy'' - xy' + y = 0$ (page 255, Problem 27)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' - xy' + y = (r^2 - r)c_0 x^{r-1} + \sum_{k=0}^{\infty}[(x + r + 1)(k + r)c_{k+1} - (k + r)c_k + c_k]x^{k+r} = 0$$

which implies

$$r^2 - r = r(r-1) = 0$$

and

$$(k + r + 1)(k + r)c_{k+1} - (k + r - 1)c_k = 0.$$

The indicial roots are $r_1 = 1 \; and \; r_2 = 0$. For $r_1 = 1$ the recurrence relation is

$$c_{k+1} = \frac{kc_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \cdots,$$

and one solution is $y_1 = c_0 x$. A second solution is

$$y_2 = x\int \frac{e^{-\int -dx}}{x^2} dx = x\int \frac{e^x}{x^2} dx = x\int \frac{1}{x^2}(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots)dx$$

$$= x\int (\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots)dx = x[-\frac{1}{x} + \ln x + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{72}x^3 + \cdots]$$

$$= x\ln x - 1 + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots.$$

The general solution on $(0, \infty)$ is

$$y = C_1 x + C_2 y_2(x).$$

7) $xy'' + (1 - x)y' - y = 0$ (page 255, Problem 29)

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting

terms, we obtain

$$xy'' + (1-x)y' - y = r^2 c_0 x^{r-1} + \sum_{k=0}^{\infty} [(k+r)(k+r-1)c_k + (k+r)c_k - (k+r)c_{k-1}]x^{k+r-1} = 0,$$

which implies $r^2 = 0$ and

$$(k+r)^2 c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r_1 = r_2 = 0$ and the recurrence relation is

$$c_k = \frac{c_{k-1}}{k}, \quad k = 1, 2, 3, \cdots$$

One solution is

$$y_1 = c_0(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots) = c_0 e^x.$$

A second solution is

$$y_2 = y_1\int \frac{e^{-\int (1/x - 1)dx}}{e^{2x}} dx = e^x \int \frac{e^x / x}{e^{2x}} dx = e^x \int \frac{1}{x} e^{-x} dx$$

$$= e^x \int \frac{1}{x}(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots)dx = e^x \int (\frac{1}{x} - 1 + \frac{1}{2}x - \frac{1}{3!}x^2 + \cdots)dx$$

$$= e^x[\ln x - x + \frac{1}{2\cdot 2}x^2 - \frac{1}{3\cdot 3!}x^3 + \cdots] = e^x \ln x - e^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!} x^n.$$

The general solution on $(0,\infty)$ is

$$y = C_1 e^x + C_2 e^x (\ln x - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!} x^n).$$

Finally, you shall be familiar with nice examples in the textbook and those problems in Homework 9.