Series

(http://www.answers.com/)

A **series** is a sum of a sequence of terms.

For example

1 + 2 + 3 + 4 + 5 + ...

Series may be finite, or *infinite*.

Infinite series

An **infinite series** is a sum of infinitely many **terms**. Such a sum can have a finite value; if it has, it is said to *converge*; otherwise it is said to *diverge*.
An infinite series is formally written as

$$\sum_{n=0}^{\infty} c_n$$

where the elements c_n are real (or complex) numbers. We say that this series **converges towards** S , or that **its value is** S, if the limit

$$\lim_{N\to\infty}\sum_{n=0}^{N} c_n = S$$

exists. If there is no such number, then the series is said to *diverge*.
The sequence of **partial sums** is defined as the sequence

$$\sum_{n=0}^{N} c_n$$

indexed by N. Then, the definition of series convergence simply says that the sequence of partial sums has limit S, as $N \to \infty$.
If the series $\sum_{n=0}^{N} c_n$ converges, then the sequence (c_n) converges to 0 for $n\to\infty$; the converse is in general not true.

Some types of infinite series

- A *geometric series* is one where each successive term is produced by multiplying the previous term by a constant number. Example:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\sum_{n=0}^{\infty}\frac{1}{2^n}.$$

- The ***harmonic series*** is the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\sum_{n=1}^{\infty}1/n.$$

The series $\sum_{n=1}^{\infty}\frac{1}{n^r}$ converges if $r > 1$ and diverges for $r \leq 1$

- An *alternating series* is a series where terms alternate signs. Example:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}.$$

Absolute convergence

The sum

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

is said to **converge absolutely** if the series of absolute values

$$\sum_{n=0}^{\infty} \left|c_n(x-a)^n\right|$$

converges. In this case, the **original series** $\sum_{n=0}^{\infty} c_n(x-a)^n$ **converges**.

Convergence tests

1. **Ratio test**: If $\left|\frac{c_{n+1}}{c_n}\right| < 1$ for all sufficiently large n, then $\sum_{n=0}^{\infty} c_n$ **converges absolutely**.
2. **Alternating series test**: A series of the form $\sum_{n=0}^{\infty} (-1)^n c_n$ (with $c_n \geq 0$) is called *alternating*. Such a series **converges** if the sequence c_n is monotone decreasing (單調遞減) and converges to 0. The converse is in general not true.

Taylor Series for Functions of One Variable

(Mathematical Handbook of Formulas and Tables, Murray R. Spiegel, 1968)

$$f(x) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

where R^n is the remainder after n terms.

If $\lim_{n\to\infty} R^n = 0$, the infinite series obtained is called the **Taylor series** for $f(x)$ about $x = a$.

If $a = 0$ the series is often called a **Maclaurin series**. These series, often called **power series,** generally converge for all values of x in some interval called the **interval of convergence** and diverge for all x outside this interval.

Power Series

http://ltcconline.net/greenl/courses/107/Series/pow.htm

Definition of a Power Series

Let $f(x)$ be the function represented by the series $f(x)=\sum_{n=0}^{\infty} c_n x^n$

Then $f(x)$ is called a *power series* function.

More generally, if $f(x)$ is represented by the series $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$

Then we call $f(x)$ a *power series centered at x = a*. The domain of f(x) is called the ***Interval of Convergence*** and half the length of the domain is called the ***Radius of Convergence.***

Ratio Test

Convergence of power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ can often be determined by the ratio test. Suppose $c_n \neq 0$ for all n, and that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

If $L<1$ the series converges absolutely,
If $L>1$ the series diverges, and
If $L=1$ the ratio test is inclusive.

Radius of Convergence

Every power series has a radius of convergence R. If $R>0$, then a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a|<R$ and diverges for $|x-a|>R$.

Interval of Convergence

To find the interval of convergence we follow the three steps:

1) Use the **ratio test** to find the interval where the series is absolutely convergent
2) Plug in the **left endpoint** to see if it converges at the left endpoint
3) Plug in the **right endpoint** to see if it converges at the right endpoint

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Example1: Find the radius of convergence of $f(x)=\sum_{n=0}^{\infty}\frac{(x-3)^n}{2^n}$

Solution: We apply the Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}}\frac{2^n}{(x-3)^n}\right|=\lim_{n\to\infty}\left|\frac{x-3}{2}\right|$$

➔ $\left|\frac{x-3}{2}\right|<1$

➔ $|x-3|<2$ ➔ $1<x<5$

➔ $\frac{1}{2}(5-1)=2$ the **radius of convergence** is $R=2$

Example2:

Find the **interval of convergence** for the previous example: $f(x)=\sum_{n=0}^{\infty}\frac{(x-3)^n}{2^n}$

Solution:

1. We have already done this step and found that the series converges absolutely
   for 1 < x < 5
2. We plug in x = 1 to get $f(1)=\sum_{n=0}^{\infty}\frac{(1-3)^n}{2^n}=\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n$
   This series diverges by the limit test.
3. We plug in x = 5 to get $f(5)=\sum_{n=0}^{\infty}\frac{(5-3)^n}{2^n}=\sum_{n=0}^{\infty}\frac{(2)^n}{2^n}=\sum_{n=0}^{\infty}1^n$

   This series also **diverges** by the limit test.

Hence the endpoints are not included in the interval of convergence. We can conclude that the interval of convergence is $1<x<5$ or $(1,\ 5)$

Example3:

Find the **interval of convergence** for $f(x)=\sum_{n=0}^{\infty}\dfrac{(x-3)^n}{2^n n}$

Solution:

1. We apply the Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}\frac{2^n n}{(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\left|\frac{n}{2(n+1)}\right| = \frac{1}{2}|x-3|$$

The series converges absolutely for $\frac{1}{2}|x-3|<1$ or $|x-3|<2$ or $1<x<5$

➔ $\frac{1}{2}(5-1)=2$ the **radius of convergence** is $R=2$

The series diverges for $|x-3|>2$ ➔ $x>5$ or $x<1$

2. (1)We plug in x = 1 to get $f(1)=\sum_{n=0}^{\infty}\dfrac{(1-3)^n}{2^n n}=\sum_{n=0}^{\infty}\dfrac{(-2)^n}{2^n n}=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n}$

This series converges by the **alternating series test**.

(2)We plug in x = 5 to get $f(5)=\sum_{n=0}^{\infty}\dfrac{(5-3)^n}{2^n n}=\sum_{n=0}^{\infty}\dfrac{(2)^n}{2^n n}=\sum_{n=0}^{\infty}\dfrac{1}{n}$

This series is the divergent **harmonic series**.

Hence the interval of convergence is $1\le x<5$ or $[1,5)$.

Differentiation and Integration of Power Series

Since a power series is a function, it is natural to ask if the function is continuous, differentiable or integrable. The following theorem answers this question.

Theorem

Suppose that a function is given by the power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

and that the interval of convergence is (a - R, a + R) (plus possible endpoints) then $f(x)$ is continuous, differentiable, and integrable on that interval (not necessarily including the endpoints). In other words

$$\frac{d\ f(x)}{dx} = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} c_n (x-a)^n = \sum_{n=0}^{\infty} n c_n (x-a)^n$$

and

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

Furthermore, the **radius of convergence** for the derivative and integral is R.

Example: Consider the series $f(x) = \sum_{n=0}^{\infty} x^n$

this series converges for $|x| < 1$, the **center of convergence** is 0 and the **radius** is 1.

By the above theorem, $f'(x) = \sum_{n=1}^{\infty} n x^{n-1}$ has **center of convergence** 0 and **radius of convergence** 1 also.

We can also say that $\int f(x)dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$

also has **center of convergence** 0 and **radius of convergence** 1.