**Definition 3.2 Wronskian**

Suppose each of the functions $f_1(x), f_2(x),..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_{2,...,} f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \quad (1)$$

Where the prime denote derivatives, is called the **Wronskian** of the functions.

以上行列式定義函數 $f_1(x), f_2(x),..., f_n(x)$ 的 **Wronskian**

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Let $y_1$ and $y_2$ be solutions of the homogeneous linear second order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad (2)$$

Thus,

$$\Rightarrow y_1'' = -\frac{a_1(x)}{a_2(x)}y_1' - \frac{a_0(x)}{a_2(x)}y_1 \quad (3)$$

$$\Rightarrow y_2'' = -\frac{a_1(x)}{a_2(x)}y_2' - \frac{a_0(x)}{a_2(x)}y_2 \quad (4)$$

假如 $y_1$ 及 $y_2$ 爲齊次、線性之二階微分方程式之解，則藉由式(3)~(4)的表示式可進行以下 **Wronskian** 性質之推導。

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$$\Rightarrow W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \quad (5)$$

$$\begin{aligned} W'(y_1, y_2) &= y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2 \\ &= y_1 y_2'' - y_1'' y_2 \\ &= y_1\left[-\frac{a_1}{a_2}y_2' - \frac{a_0}{a_2}y_2\right] - y_2\left[-\frac{a_1}{a_2}y_1' - \frac{a_0}{a_2}y_1\right] \\ &= -\frac{a_1}{a_2}y_1 y_2' - \frac{a_1}{a_2}y_1 y_2 + \frac{a_1}{a_2}y_1' y_2 + \frac{a_0}{a_2}y_1 y_2 \\ &= -\frac{a_1}{a_2}y_1 y_2' + \frac{a_1}{a_2}y_1' y_2 \\ &= -\frac{a_1}{a_2}\left[y_1 y_2' - y_1' y_2\right] = -\frac{a_1}{a_2}W(y_1, y_2) \end{aligned} \quad (6)$$

➔ $W'(y_1,y_2)+\frac{a_1}{a_2}W(y_1,y_2)=0$ (7)

➔integrating factor $\mu=e^{\int\frac{a_1(x)}{a_2(x)}dx}$ (8)

➔ $\frac{d}{dx}\left[e^{\int\frac{a_1(x)}{a_2(x)}dx}W(y_1,y_2)\right]=0$ (9)

➔ $e^{\int\frac{a_1(x)}{a_2(x)}dx}W(y_1,y_2)=C$ (10)

➔ $W(y_1,y_2)=Ce^{-\int\frac{a_1(x)}{a_2(x)}dx}$ (11)

Note that $e^{-\int\frac{a_1(x)}{a_2(x)}dx}\neq 0$ for any value of $x$

自然指數函數不為零的性質，可作為以下 **Wronskian** 性質之推論

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If $W(y_1,y_2)=0$ for even one point

Let $W(y_1(x_0),y_2(x_0))=0$

➔ $C=0$ ➔ $W(y_1,y_2)=0$ for all $x$

Otherwise $W(y_1,y_2)\neq 0$

所以若存在(即使僅區間內一點) **Wronskian** $W(y_1,y_2)=0$ ，則可推論區間內 $W(y_1,y_2)=0$ ，否則區間內 $W(y_1,y_2)\neq 0$ 。即為以下之定理陳述：

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**Theorem: Wronskian Test**

Let $y_1$ and $y_2$ be solutions of $a_2(x)y''+a_1(x)y'+a_0(x)y=0$ on an interval $I$ . Then

1) Either $W(y_1,y_2)=0$ for all $x$ in the interval $I$ , or $W(y_1,y_2)\neq 0$ for all $x$ in the interval $I$ .
2) $y_1$ and $y_2$ are **linearly independent** on the interval $I$ if and only if $W(y_1,y_2)\neq 0$ on the interval $I$ .