1) Find the general solution of the given second-order differential equation.

$y'' + y = \sin(x)$ (page 136, Problem 3)

Step1:

Find y_c

➔ $y_c = e^{mx}$, $y'_c = me^{mx}$, $y''_c = m^2 e^{mx}$

➔ $m^2 + 1 = 0$, $m = \pm\ i$

➔ $y_c = c_1\cos(x) + c_2\sin(x)$

Step2:

Find y_p

a)此為一常係數之非齊式線性微分方程式，且非齊式項為$\sin(x)$函數

➔待定係數法適用之

➔ $y_p = A\sin(x) + B\cos(x)$, modified➔ $y_p = Ax\sin(x) + Bx\cos(x)$

(因為 $y_c = c_1\cos(x) + c_2\sin(x)$)

➔ $y'_p = A\sin(x) + Ax\cos(x) + B\cos(x) - Bx\sin(x)$

➔ $y''_p = 2A\cos(x) - 2B\sin(x) - Ax\sin(x) - Bx\cos(x)$

➔ $y''_p + y_p = 2A\cos(x) - 2B\sin(x) = \sin(x)$

➔ $A = 0,\ B = -\frac{1}{2}$

➔ $y_p = -\frac{1}{2}x\cos(x)$ (1)

➔ $y = y_c + y_p = c_1\cos(x) + c_2\sin(x) - \frac{1}{2}x\cos(x)$ (2)

b)參數變易法

$y_1 = \cos(x)$, $y_2 = \sin(x)$ ➔ $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix} = \begin{vmatrix} 0 & \sin(x) \\ \sin(x) & \cos(x) \end{vmatrix} = -\sin^2(x)$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sin(x) \end{vmatrix} = \cos(x)\sin(x)$$

➔ $y_p = u_1 y_1 + u_2 y_2$

➔ $u'_1 = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} = -\sin^2(x)$ ➔ $u'_1 = -\sin^2(x) = \frac{\cos(2x)-1}{2}$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$u_1 = \frac{\sin(2x)-2x}{4}$$

➔ $u'_2 = \frac{W_2}{W} = \frac{y_1 f(x)}{W} = \cos(x)\sin(x)$ ➔ $u'_2 = \cos(x)\sin(x) = \frac{\sin(2x)}{2}$

$$\cos(x)\sin(x) = \frac{\sin(2x)}{2}$$

$$u_2 = -\frac{\cos(2x)}{4}$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

➔ $y_p = u_1 y_1 + u_2 y_2 = \frac{\cos(x)\sin(2x) - 2x\cos(x)}{4} - \frac{\sin(x)\cos(2x)}{4} = \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$

➔ $y_p = \frac{\sin(x)}{2} - \frac{x\cos(x)}{2}$ (3)

General solution

$$y = y_c + y_p = c_1\cos(x) + c_2\sin(x) + \frac{\sin(x)}{2} - \frac{x\cos(x)}{2}$$

➔ $= c_1\cos(x) + \left(c_2 + \frac{1}{2}\right)\sin(x) - \frac{x\cos(x)}{2}$ (4)

$$= c_1\cos(x) + C_2\sin(x) - \frac{x\cos(x)}{2}$$

所以，雖然所求之特解的型式(1)、(3)有所不同，但通解的型式(2)、(4)可說是相同的。因此，仍無違背定理 3.1 的唯一解之存在性。