Engineering Mathematics I---Quiz-8s

1) Find the radius of convergence and interval of convergence for the given power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$$

ANS $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n / n}\right| = \lim_{n\to\infty}\frac{2n}{n+1}|x| = 2|x|$

The series is absolutely convergent for $2|x|<1\, or\, |x|<1/2$. At $x=-1/2$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by the alternating series test. At $x=1/2$, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which diverges. Thus, the given series converges on $[-1/2, 1/2)$.

2) Use the power series method to solve the given initial-value problem.

$$(x-1)y'' - xy' + y = 0, \quad y(0) = -2, \quad y'(0) = 6$$

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x-1)y'' - xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_k x^k + \sum_{k=0}^{\infty}c_k x^k$$

$$= -2c_2 + c_0 + \sum_{k=1}^{\infty}[-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k]x^k = 0.$$

Thus

$$-2c_2 + c_0 = 0$$
$$-(k+2)(k+1)c_{k+2} + (k-1)kc_{k+1} - (k-1)c_k = 0$$

and

$$c_2 = \frac{1}{2}c_0$$

$$c_{k+2} = \frac{kc_{k+1}}{k+2} - \frac{(k-1)c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \cdots$$

Choosing $c_0 = 1 \; and \; c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{6}, \quad c_4 = 0$$

and so on. For $c_0 = 0 \; and \; c_1 = 1$ we obtain $c_2 = c_3 = c_4 = \cdots = 0.$ Thus,

$$y = C_1(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots) + C_2 x$$

and

$$y' = C_1(x + \frac{1}{2}x^2 + \cdots) + C_2.$$

The initial conditions imply $C_1 = -2 \; and \; C_2 = 6,$ so

$$y = -2(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots) + 6x = 8x - 2e^x.$$

3) In this problem $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Using the method of Frobenius to obtain at least one series solutions about $x = 0$. Form the general solution on $(0, \; \infty)$.

$$xy'' - xy' + y = 0$$

ANS Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' - xy' + y = (r^2 - r)c_0 x^{r-1} + \sum_{k=0}^{\infty}[(x + r + 1)(k + r)c_{k+1} - (k + r)c_k + c_k]x^{k+r} = 0$$

which implies

$$r^2 - r = r(r-1) = 0$$

and

$$(k + r + 1)(k + r)c_{k+1} - (k + r - 1)c_k = 0.$$

The indicial roots are $r_1 = 1 \; and \; r_2 = 0.$ For $r_1 = 1$ the recurrence relation is

$$c_{k+1} = \frac{kc_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \cdots,$$

and one solution is $y_1 = c_0 x.$ A second solution is

$$y_2 = x\int \frac{e^{-\int -dx}}{x^2}dx = x\int \frac{e^x}{x^2}dx = x\int \frac{1}{x^2}(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots)dx$$

$$= x\int (\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots)dx = x[-\frac{1}{x} + \ln x + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{72}x^3 + \cdots]$$

$$= x \ln x - 1 + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots.$$

The general solution on $(0, \infty)$ is

$$y = C_1 x + C_2 y_2(x).$$