1, Find the general solution using the method of variation of parameters

$y'' + 9y = 12\sec(3x)$

Sol：

(1) 求 y_h :

$y'' + 9y = 0$

set $y = e^{sx}$

$s^2 e^{st} + 9 e^{st} = 0$

$s = \pm 3i$

$y_h = c_1 \cos[3x] + c_2 \sin[3x]$

(2) 求 y_p :

set $y_1 = \cos[3x]$, $y_2 = \sin[3x]$

$|w| = \begin{pmatrix} \cos[3x] & \sin[3x] \\ -3\sin[3x] & 3\cos[3x] \end{pmatrix} = 3(\cos^2[3x] + \sin^2[3x]) = 3$

$v' = \frac{y_1 * f[x]}{|w|} = \frac{\cos[3x] * 12\sec[3x]}{3} = 4$

$u' = -\frac{y_2 * f[x]}{|w|} = \frac{\sin[3x] * 12\sec[3x]}{3} = -4\tan[3x]$

$u = \frac{4}{3} \ln[\cos[3x]]$

$v = 4x$

$y_p = u y_1 + v y_2 = \frac{4}{3} \cos[3x] * \ln[\cos[3x]] + 4x * \sin[3x]$

$y = y_h + y_p = c_1 \cos[3x] + c_2 \sin[3x] + \frac{4}{3} \cos[3x] * \ln[\cos[3x]] + 4x * \sin[3x]$

$= \left(c_1 + \frac{4}{3} \ln[\cos[3x]]\right) \cos[3x] + (c_2 + 4x) \sin[3x]$

2, Find the general solution using the method of undetermined coefficients

$y'' + 4y' + 4y = 7x - 3\cos(2x) + 5x e^{-2x}$

Sol :

(1) 求y_h :

$y'' + 4y' + 4y = 0$

set $y = e^{sx}$

$s^2 e^{sx} + 4s e^{sx} + 4 e^{sx} = 0$

$s = -2$ (重根)

$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

(2) 求y_p

PART (A) :

$y'' + 4y' + 4y = 7x$

set $y_{p1} = ax + b$, $(y_{p1})' = a$, $(y_{p1})'' = 0$

$4a + 4ax + 4b = 7x$

$\begin{cases} 4a = 7x \\ 4a + 4b = 0 \end{cases} = \begin{cases} a = \frac{7}{4} \\ b = \frac{-7}{4} \end{cases}$

$y_{p1} = \frac{7}{4} x - \frac{7}{4}$

PART (B) :

$y'' + 4y' + 4y = -3\cos(2x)$

set $y_{p2} = c * \cos[2x] + d * \sin[2x]$, $(y_{p2})' = -2c * \sin[2x] + 2d * \cos[2x]$

$(y_{p2})'' = -4c * \cos[2x] - 4d * \sin[2x]$

$-4c * \cos[2x] - 4d * \sin[2x] - 8c * \sin[2x] +$

$8d * \cos[2x] + 4c * \cos[2x] + 4d * \sin[2x] = -3\cos(2x)$

$-8c * \sin[2x] + 8d * \cos[2x] = -3\cos(2x)$

$\begin{cases} 8d = -3 \\ -8c = 0 \end{cases} = \begin{cases} c = 0 \\ d = \frac{-3}{8} \end{cases}$

$y_{p2} = \frac{-3}{8} \sin[2x]$

PART (C) :

$y'' + 4y' + 4y = 5x e^{-2x}$

set $y_{p3} = (fx^3 + gx^2) e^{-2x}$, $(y_{p3})' = (3fx^2 + 2gx) e^{-2x} - 2(fx^3 + gx^2) e^{-2x}$,

$(y_{p3})'' =$

$(6fx + 2g) e^{-2x} - 2(3fx^2 + 2gx) e^{-2x} - 2(3fx^2 + 2gx) e^{-2x} + 4(fx^3 + gx^2) e^{-2x}$

$= (6fx + 2g) e^{-2x} - 4(3fx^2 + 2gx) e^{-2x} + 4(fx^3 + gx^2) e^{-2x}$

$(y_{p3})'' + 4(y_{p3})' + 4y_{p3} = (6fx + 2g) e^{-2x} = 5x e^{-2x}$

$\begin{cases} 6f = 5 \\ 2g = 0 \end{cases} = \begin{cases} f = \frac{5}{6} \\ g = 0 \end{cases}$

$y_{p3} = \frac{5}{6} x^3 e^{-2x}$

$y = y_h + y_{p1} + y_{p2} + y_{p3}$

$= c_1 e^{-2x} + c_2 x e^{-2x} - \frac{3}{8} \sin[2x] + \frac{5}{6} x^3 e^{-2x}$

3, Show that $y_1(x) = x^2$ and $y_2(x) = x - 1$ are solutions of $(x^2 - 2x) y'' + 2(1 - x) y' + 2y = 0$, use this to find the general solution of $(x^2 - 2x) y'' + 2(1 - x) y' + 2y = 6(x^2 - 2x)^2$

Sol :

$(x^2 - 2x) y_1'' + 2(1 - x) y_1' + 2y_1 = 2(x^2 - 2x) + 2(1 - x)(2x) + 2(x^2) = 0$

$(x^2 - 2x) y_2'' + 2(1 - x) y_2' + 2y_2 = (x^2 - 2x) * 0 + 2(1 - x)(1) + 2(x - 1) = 0$

$y_h = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 (x - 1)$

set $y_p = ax^7 + bx^6 + cx^5 + dx^4 + ex^3$

$y_p' = 7ax^6 + 6bx^5 + 5cx^4 + 4dx^3 + 3ex^2$

$y_p'' = 42ax^5 + 30bx^4 + 20cx^3 + 12dx^2 + 6ex$

$(x^2 - 2x) y_p'' + 2(1 - x) y_p' + 2y_p =$

$(x^2 - 2x)(42ax^5 + 30bx^4 + 20cx^3 + 12dx^2 + 6ex) +$

$2(1 - x)(7ax^6 + 6bx^5 + 5cx^4 + 4dx^3 + 3ex^2) + 2(ax^7 + bx^6 + cx^5 + dx^4 + ex^3)$

$30ax^7 - (70a - 20b)x^6 - (48b - 12c)x^5 - (30c - 6d)x^4 - (16d - 2e)x^3 - 6ex^2 =$

$6(x^2 - 2x)^2 = 6x^4 - 24x^3 + 24x^2$

$30a = 0$	$a = 0$
$70a - 20b = 0$	$b = 0$
$48b - 12c = 0$	$c = 0$
$30c - 6d = -6$	$d = 1$
$16d - 2e = 24$	$e = -4$
$6e = -24$	

$y_p = x^4 - 4x^3$

$y = y_h + y_p = c_1 x^2 + c_2 (x - 1) + x^4 - 4x^3$

4, Find the general solution of the differential equation or the solution of the initial value problem

$yy'' - 2(y')^2 = 0$

Sol :

set $y' = u$, $y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$

$y u \frac{du}{dy} - 2u^2 = 0$

$2u = y \frac{du}{dy}$

$\int \frac{1}{2u} du = \int \frac{1}{y} dy$

$$\frac{1}{2}\ln[u] + c = \ln[y]$$

$$u = y' = a\,y^2$$

$$\int \frac{1}{a\,y^2}\,dy = \int dx$$

$$x + b = \frac{-1}{a}\,y^{-1}$$

$$y = \frac{1}{-ax - ab} = \frac{1}{c_1\,x + c_2}$$