1, Using Theorem 7.10 to determine wether the matrix is nonsingular. If it is, use Theorem 7.11 to find its inverse.

$$\begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}$$

Sol :

$$A = \begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}$$

$$\text{Det}\left[\begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}\right] = -24 - 6 + 3 + 56 = 29 \neq 0$$

Using Theorem 7.10

$$\text{Det}\left[\begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}\right] \neq 0$$

So $\begin{pmatrix} 0 & -4 & 3 \\ 2 & -1 & 6 \\ 1 & -1 & 7 \end{pmatrix}$ is nonsingular

$$b_{11} = \frac{1}{29} \begin{pmatrix} -1 & 6 \\ -1 & 7 \end{pmatrix} = \frac{-1}{29}$$

$$b_{12} = \frac{1}{29} (-1) \begin{pmatrix} -4 & 3 \\ -1 & 7 \end{pmatrix} = \frac{25}{29}$$

$$b_{13} = \frac{1}{29} \begin{pmatrix} -4 & 3 \\ -1 & 6 \end{pmatrix} = \frac{-21}{29}$$

$$b_{21} = \frac{1}{29} (-1) \begin{pmatrix} 2 & 6 \\ 1 & 7 \end{pmatrix} = \frac{-8}{29}$$

$$b_{22} = \frac{1}{29} \begin{pmatrix} 0 & 3 \\ 1 & 7 \end{pmatrix} = \frac{-3}{29}$$

$$b_{23} = \frac{1}{29} (-1) \begin{pmatrix} 0 & 3 \\ 2 & 6 \end{pmatrix} = \frac{6}{29}$$

$$b_{31} = \frac{1}{29} \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} = \frac{-1}{29}$$

$$b_{32} = \frac{1}{29} (-1) \begin{pmatrix} 0 & -4 \\ 1 & -1 \end{pmatrix} = \frac{-4}{29}$$

$$b_{33} = \frac{1}{29} \begin{pmatrix} 0 & -4 \\ 2 & -1 \end{pmatrix} = \frac{8}{29}$$

$$A^{-1} = \begin{pmatrix} \frac{-1}{29} & \frac{25}{29} & \frac{-21}{29} \\ \frac{-8}{29} & \frac{-3}{29} & \frac{6}{29} \\ \frac{-1}{29} & \frac{-4}{29} & \frac{8}{29} \end{pmatrix}$$

2, Compute the indicated power of the matrix, using the idea of Porblem 14.

Sol :

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} ;\ A^{43}$$

$$(A - \lambda I)\ \{x\} = \{0\}$$

$$\text{Det}\ (A - \lambda I) = 0$$

$$\begin{pmatrix} -\lambda & -2 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 2 = 0$$

$$\lambda = \pm \sqrt{2}\ i$$

(1) $\lambda = \sqrt{2}\ i$

$$\begin{pmatrix} -\sqrt{2}\,i & -2 \\ 1 & -\sqrt{2}\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_1 = \begin{pmatrix} \sqrt{2}\,i \\ 1 \end{pmatrix}$$

(2) $\lambda = -\sqrt{2}\,i$

$$\begin{pmatrix} \sqrt{2}\,i & -2 \\ 1 & \sqrt{2}\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 = \begin{pmatrix} -\sqrt{2}\,i \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \sqrt{2}\,i & -\sqrt{2}\,i \\ 1 & 1 \end{pmatrix}$$

$$A = P \begin{pmatrix} \sqrt{2}\,i & 0 \\ 0 & -\sqrt{2}\,i \end{pmatrix} P^{-1}$$

$$A^{43} = P \begin{pmatrix} \left(\sqrt{2}\right)^{43} (i)^{43} & 0 \\ 0 & \left(-\sqrt{2}\right)^{43} (i)^{43} \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} \left(\sqrt{2}\right)^{43} (i)^{43} & 0 \\ 0 & \left(-\sqrt{2}\right)^{43} (i)^{43} \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 0 & 2^{22} \\ -2^{21} & 0 \end{pmatrix}$$

3, Find the eigenvalues of the matrix and for each eigenvalue a corresponding eigenvector.
Check that eigenvectors associated with distinct eigenvalues are orthogonal.
Find an orthogonal matrix that diagonalizes the Matrix.

(1) $\begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$

(2) $\begin{pmatrix} 5 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

Sol :

(1)

$$A = \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$$

$(A - \lambda I)\{x\} = \{0\}$

Det $(A - \lambda I) = 0$

$$\begin{pmatrix} 6-\lambda & 1 \\ 1 & 4-\lambda \end{pmatrix} = 0$$

$(6-\lambda)(4-\lambda) - 1 = 0$

$\lambda = 5 \pm \sqrt{2}$

(a) $\lambda = 5 + \sqrt{2}$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 - \left(1+\sqrt{2}\right) x_2 = 0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_a = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$ 單位化 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_a = \begin{pmatrix} 0.9239 \\ 0.3827 \end{pmatrix}$

(b) $\lambda = 5 - \sqrt{2}$

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - \left(1-\sqrt{2}\right) x_2 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_b = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \text{ 單位化 } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_b = \begin{pmatrix} -0.3827 \\ 0.9239 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_a \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_b = 0$$

orthogonal matrix

$$(\{x\}_a, \{x\}_b) = \begin{pmatrix} 0.9239 & -0.3827 \\ 0.3827 & 0.9239 \end{pmatrix}$$

(2)

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$(A - \lambda I) \{x\} = \{0\}$$

$$\text{Det } (A - \lambda I) = 0$$

$$\begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 (5-\lambda) + 4\lambda = 0$$

$$\lambda = \frac{5 \pm \sqrt{41}}{2}, 0$$

(a) $\lambda = \frac{5+\sqrt{41}}{2}$

$$\begin{pmatrix} \frac{5-\sqrt{41}}{2} & 0 & 2 \\ 0 & -\frac{5+\sqrt{41}}{2} & 0 \\ 2 & 0 & -\frac{5+\sqrt{41}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_a = \begin{pmatrix} \frac{5+\sqrt{41}}{4} \\ 0 \\ 1 \end{pmatrix} \text{ 單位化 } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_a = \begin{pmatrix} 0.9436 \\ 0 \\ 0.3310 \end{pmatrix}$$

(b) $\lambda = \frac{5-\sqrt{41}}{2}$

$$\begin{pmatrix} \frac{5+\sqrt{41}}{2} & 0 & 2 \\ 0 & -\frac{5-\sqrt{41}}{2} & 0 \\ 2 & 0 & -\frac{5-\sqrt{41}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_b = \begin{pmatrix} \frac{5-\sqrt{41}}{4} \\ 0 \\ 1 \end{pmatrix} \text{ 單位化 } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_b = \begin{pmatrix} -0.3310 \\ 0 \\ 0.9436 \end{pmatrix}$$

(c) $\lambda = 0$

$$\begin{pmatrix} 5 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_a \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_b = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_b \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_c = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_a \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_c = 0$$

orthogonal matrix

$$(\{X\}_a, \{X\}_b, \{X\}_c) = \begin{pmatrix} 0.9436 & -0.3310 & 0 \\ 0 & 0 & 1 \\ 0.3310 & 0.9436 & 0 \end{pmatrix}$$