1, Find the eigenvalues of the matrix and for each eigenvalue, a corresponding eigenvector.
Check that eigenvectors associated with distinct eigenvalues are orthogonal.
Find an otrhogonal matrix that diagonalizes the matrix. (P362, Problem 11)

Sol :

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$(A - \lambda I)\{x\} = \{0\}$

$\text{Det}\,(A - \lambda I) = 0$

$$\begin{pmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -2 & 0 \\ 0 & -2 & 1-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = 0$$

$$-\lambda \begin{pmatrix} 1-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = 0$$

$-\lambda[-\lambda\,(1-\lambda)^2 + 4\lambda] = 0$

$\lambda^2\,(1-\lambda)^2 - 4\lambda^2 = 0$

$\lambda^2\,(1+\lambda)\,(\lambda-3) = 0$

$\lambda = 3, -1, 0, 0$

(1) $\lambda = 3$

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$-3x_1 = 0$

$\Longrightarrow\ -2x_2 - 2x_3 = 0$

$-3x_4 = 0$

$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

(2) $\lambda = -1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 0$

$\Longrightarrow\ 2x_2 - 2x_3 = 0$

$x_4 = 0$

$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

(2) $\lambda = 0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow\ x_2 - 2x_3 = 0$

$-2x_2 + x_3 = 0$

令$x_1 = 1,\ x_2 = 0,\ x_4 = 0$

$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

令$x_1 = 0,\ x_2 = 0,\ x_4 = 1$

$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

特徵向量單位化 (使特徵向量長度 = 1)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_1 = \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

orthongonal matrix

$$(\{x\}_1,\ \{x\}_2,\ \{x\}_3,\ \{x\}_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2, Find the standard form of the quadratic form.

$-2x_1 x_2 + 2x_3^2$

Sol :

$-2x_1 x_2 + 2x_3^2$

$= 0x_1^2 + 0x_2^2 + 2x_3^2 - 2x_1 x_2 + 0x_2 x_3 + 0x_1 x_3$

$$= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$(A - \lambda I)\{x\} = \{0\}$

$\text{Det}(A - \lambda I) = 0$

$$\begin{pmatrix} -\lambda & -1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$$

$\lambda^2 (2-\lambda) - (2-\lambda) = 0$

$\lambda = -1, 1, 2$

(1) $\lambda = -1$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \text{ 單位化 } \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

(2) $\lambda = 1$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \text{ 單位化 } \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

(3) $\lambda = 2$

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

standard form of the quadratic form

$X^T A X = Y^T (Q^T A Q) Y$

$$= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$= -y_1^2 + y_2^2 + 2 y_3^2$

3, Use the principal axis theorem to analyze the conic

$3 x_1^2 + 5 x_1 x_2 - 3 x_2^2 = 5$

Sol :

$X^T A X = 5$

$$A = \begin{pmatrix} 3 & \frac{5}{2} \\ \frac{5}{2} & -3 \end{pmatrix}$$

$(A - \lambda I) \{x\} = \{0\}$

$\text{Det}(A - \lambda I) = 0$

$$\begin{pmatrix} 3-\lambda & \frac{5}{2} \\ \frac{5}{2} & -3-\lambda \end{pmatrix} = 0$$

$$(9 - \lambda^2) + \frac{25}{4} = 0$$

$\lambda = \pm \frac{\sqrt{61}}{2}$

standard form of the quadratic form

$X^T A X = 5$

$Y^T (Q^T A Q) Y = 5$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} \frac{\sqrt{61}}{2} & 0 \\ 0 & -\frac{\sqrt{61}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 5$$

$$\frac{\sqrt{61}}{2} y_1^2 - \frac{\sqrt{61}}{2} y_2^2 = 5$$

是個雙曲線

4, Revisit Example Problem 8.17 and calculate the angle of the y1 axis relative to the x1 axis, as shown in Figure 8.2

Sol :

$$4 x_1^2 - 3 x_1 x_2 + 2 x_2^2 = 8$$

$$A = \begin{pmatrix} 4 & -\frac{3}{2} \\ -\frac{3}{2} & 2 \end{pmatrix}$$

$(A - \lambda I) \{x\} = \{0\}$

$\text{Det} (A - \lambda I) = 0$

$$\begin{pmatrix} 4 - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & 2 - \lambda \end{pmatrix} = 0$$

$$(4 - \lambda)(2 - \lambda) - \frac{9}{4} = 0$$

$$\lambda = \frac{6 \pm \sqrt{13}}{2}$$

(1) $\lambda = \frac{6 - \sqrt{13}}{2}$

$$\begin{pmatrix} \frac{2+\sqrt{13}}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{-2+\sqrt{13}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

令$x_2 = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_1 = \begin{pmatrix} \frac{-2+\sqrt{13}}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5352 \\ 1 \end{pmatrix} \text{單位化} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_1 = \begin{pmatrix} 0.4719 \\ 0.8817 \end{pmatrix}$$

(1) $\lambda = \frac{6 + \sqrt{13}}{2}$

$$\begin{pmatrix} \frac{2-\sqrt{13}}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{-2-\sqrt{13}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

令$x_2 = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 = \begin{pmatrix} \frac{-2-\sqrt{13}}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} -1.8685 \\ 1 \end{pmatrix} \text{單位化} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 = \begin{pmatrix} -0.8817 \\ 0.4719 \end{pmatrix}$$

$$Q = (\{x\}_1 \{x\}_2) = \begin{pmatrix} 0.4719 & -0.8817 \\ 0.8817 & 0.4719 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.4719 & -0.8817 \\ 0.8817 & 0.4719 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.4719 & 0.8817 \\ -0.8817 & 0.4719 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$y_1 = 0.4719 x_1 + 0.8817 x_2$

$$\theta = \tan^{-1} \left(\frac{0.8817}{0.4719} \right) = 61.8436$$