1, $y - xy' = 0$

(a) $y - x\frac{dy}{dx} = 0$

$ydx - xdy = 0$

令$M = y,\ N = -x$

$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$

So not exact

(b) 原式：$\frac{\partial(-\mu x)}{\partial x} = \frac{\partial(\mu y)}{\partial y}$

因爲只有x函數，所以 $\frac{\partial\mu}{\partial y} = 0$

得 $-\frac{\partial\mu}{\partial x}x = 2\mu$

$\int -\frac{1}{x}dx = \int \frac{1}{2\mu}d\mu$

$\mu = x^{-2}$

(c) 因爲只有y函數，所以 $\frac{\partial\mu}{\partial x} = 0$

$-2\mu = \frac{\partial\mu}{\partial y}y$

$\int \frac{1}{y}dx = -\int \frac{1}{2\mu}d\mu$

$\mu = y^{-2}$

(d) $x^a y^b y - x^a y^b xy' = 0$

$(x^a y^{b+1})dx - (x^{a+1} y^b)dy = 0$

令$M = x^a y^{b+1},\ N = x^{a+1} y^b$

$\frac{\partial M}{\partial y} = (b+1)x^a y^b$

$\frac{\partial N}{\partial x} = -(a+1)x^a y^b$

$\therefore a + b = -2$ for $\eta(x, y) = x^a y^b$

2, (a)

sol:

set $p(x) = x,\ R(x) = x,\ \alpha = 2$

$\therefore v = y^{-1},\ y = \frac{1}{v},\ y' = \frac{-1}{v^2}v'$

$\frac{-v'}{v^2} + \frac{x}{v} = x * \frac{1}{v^2}$

$I = e^{-\int x dx} = e^{\frac{-1}{2}x^2}$

$e^{\frac{-1}{2}x^2}v' - x * e^{\frac{-1}{2}x^2}v = -x * e^{\frac{-1}{2}x^2}$

$\left(e^{\frac{-1}{2}x^2}v\right)' = -x * e^{\frac{-1}{2}x^2}$

$\int\left(e^{\frac{-1}{2}x^2}v\right)' = \int -x * e^{\frac{-1}{2}x^2}dx$

$v e^{\frac{-1}{2}x^2} = e^{\frac{-1}{2}x^2} + c$

$\therefore v = 1 + ce^{\frac{1}{2}x^2}$, $y = \frac{1}{1 + ce^{\frac{1}{2}x^2}}$

(b)

sol: $(x - 2y)y' = 2x - y$

$(-2x + y)dx + (x - 2y)dy = 0$

$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$

$$\phi(x, y) = \int (-2x + y)\,dx + g(y)$$
$$= -x^2 + xy + g(y)$$
$$\phi(x, y) = \int (x - 2y)\,dy + f(x)$$
$$= xy - y^2 + f(x)$$
$$\therefore \phi(x, y) = -x^2 + xy - y^2$$
$$-x^2 + xy - y^2 = c$$

(c)

sol :

set $y = e^x + \frac{1}{z}$ (e^x is particular solution)

$$y' = e^x - \frac{z'}{z^2}$$
$$e^x - \frac{z'}{z^2} = -e^{-x}\left(e^x + \frac{1}{z}\right)^2 + \left(e^x + \frac{1}{z}\right) + e^x$$
$$-\frac{z'}{z^2} = -e^{-x}\left(e^{2x} + \frac{2e^x}{z} + \frac{1}{z^2}\right) + \left(e^x + \frac{1}{z}\right)$$
$$-\frac{z'}{z^2} = -e^x - \frac{2}{z} - \frac{e^{-x}}{z^2} + e^x + \frac{1}{z}$$
$$z' - z = e^{-x}$$
$$I = e^{\int -1\,dx} = e^{-x}$$
$$e^{-x} z' - e^{-x} z = e^{-2x}$$
$$(e^{-x} z)' = e^{-2x}$$
$$e^{-x} z = -\frac{1}{2} e^{-2x} + c$$
$$z = -\frac{1}{2} e^{-x} + ce^x$$
$$\frac{1}{z} = \frac{2e^x}{2ce^x - 1}$$
$$y = e^x + \frac{2e^x}{Ce^x - 1}$$