國立台灣海洋大學　93學年度第二學期　工程數學(二)　期末考

1. $$\frac{d^2x_1}{dt^2}=x_1+3x_2+t^2+1$$
$$\frac{d^2x_2}{dt^2}=4x_1+2x_2+t$$

(a) Find the matrix A of the system $\tilde{x}''=A\tilde{x}+\tilde{b}$. (5 cores)

(b) Find all eigenvalues and corresponding eigenvectors, and write the transition matrix P of A. (5 cores)

(c) write the general solution of the system $\tilde{x}''=A\tilde{x}$. (Hint: Let $\tilde{x}=P\tilde{y}$) (10 cores)

(d) write the general solutionof the system $\tilde{x}''=A\tilde{x}+\tilde{b}$. (20 cores)

(a) $\underset{\sim}{x}''=\begin{bmatrix}1&3\\4&2\end{bmatrix}\underset{\sim}{x}+\begin{Bmatrix}t^2+1\\t\end{Bmatrix}$

(b) $\lambda_1=-2\rightarrow \underset{\sim}{P_1}=\begin{Bmatrix}1\\-1\end{Bmatrix}$; $\lambda_2=5\rightarrow \underset{\sim}{P_2}=\begin{Bmatrix}3\\4\end{Bmatrix}$

$\therefore P=\begin{bmatrix}1&3\\-1&4\end{bmatrix}$

(c) 令 $\underset{\sim}{x}=p\underset{\sim}{y}$

$$y''=P^{-1}AP\underset{\sim}{y}=D_A\underset{\sim}{y}=\begin{bmatrix}-2&0\\0&5\end{bmatrix}\begin{Bmatrix}y_1\\y_2\end{Bmatrix}\rightarrow\begin{cases}y_1''+2y_1=0\\y_2''-5y_2=0\end{cases}\rightarrow\begin{cases}y_1=c_1\sin(\sqrt{2}t)+c_2\cos(\sqrt{2}t)\\y_2=c_3e^{\sqrt{5}t}+c_4e^{-\sqrt{5}t}\end{cases}$$

$$\underset{\sim}{x}=P\begin{Bmatrix}y_1\\y_2\end{Bmatrix}=\begin{bmatrix}1&3\\-1&4\end{bmatrix}\begin{Bmatrix}c_1\sin(\sqrt{2}t)+c_2\cos(\sqrt{2}t)\\c_3e^{\sqrt{5}t}+c_4e^{-\sqrt{5}t}\end{Bmatrix}=\begin{Bmatrix}c_1\sin(\sqrt{2}t)+c_2\cos(\sqrt{2}t)+3c_3e^{\sqrt{5}t}+3c_4e^{-\sqrt{5}t}\\-c_1\sin(\sqrt{2}t)-c_2\cos(\sqrt{2}t)+4c_3e^{\sqrt{5}t}+4c_4e^{-\sqrt{5}t}\end{Bmatrix}$$

(d) 令 $\underset{\sim}{x}=p\underset{\sim}{y}$

$$y''=P^{-1}AP\underset{\sim}{y}+P^{-1}\underset{\sim}{b}=D_A\underset{\sim}{y}+P^{-1}\underset{\sim}{b}=\begin{bmatrix}-2&0\\0&5\end{bmatrix}\begin{Bmatrix}y_1\\y_2\end{Bmatrix}+\frac{1}{7}\begin{Bmatrix}4t^2-3t+4\\t^2+t+1\end{Bmatrix}$$

$$\therefore\begin{cases}y_1''+2y_1=\frac{1}{7}(4t^2-3t+4)\\y_2''-5y_2=\frac{1}{7}(t^2+t+1)\end{cases}\rightarrow\begin{cases}y_1=c_1\sin(\sqrt{2}t)+c_2\cos(\sqrt{2}t)+\frac{2}{7}t^2-\frac{3}{2}t+\frac{12}{7}\\y_2=c_3e^{\sqrt{5}t}+c_4e^{-\sqrt{5}t}-\frac{1}{35}t^2-\frac{1}{35}t-\frac{7}{175}\end{cases}$$

$$\underset{\sim}{x}=P\begin{Bmatrix}y_1\\y_2\end{Bmatrix}=\begin{bmatrix}1&3\\-1&4\end{bmatrix}\begin{Bmatrix}c_1\sin(\sqrt{2}t)+c_2\cos(\sqrt{2}t)+\frac{2}{7}t^2-\frac{3}{2}t+\frac{12}{7}\\c_3e^{\sqrt{5}t}+c_4e^{-\sqrt{5}t}-\frac{1}{35}t^2-\frac{1}{35}t-\frac{7}{175}\end{Bmatrix}$$

$$=\begin{Bmatrix} c_1\sin(\sqrt{2}t)+c_2\cos(\sqrt{2}t)+3c_3e^{\sqrt{5}t}+3c_4e^{-\sqrt{5}t}+\frac{7}{35}t^2-\frac{111}{70}t+\frac{279}{175} \\ -c_1\sin(\sqrt{2}t)-c_2\cos(\sqrt{2}t)+4c_3e^{\sqrt{5}t}+4c_4e^{-\sqrt{5}t}-\frac{2}{5}t^2+\frac{97}{70}t-\frac{328}{175} \end{Bmatrix}$$

2. Consider the initial value problem

$$\begin{aligned} x_1' &= 2x_1 \\ x_2' &= 6x_2-4x_3 \qquad\qquad x_1(0)=1,\ \ x_2(0)=-1,\ \ x_3(0)=2 \\ x_3' &= 4x_2-2x_3 \end{aligned}$$

(a) write the matrix A of the system $X' = AX$. (2 scores)

(b) find the eigenvalues of the matrix A. (3 scores)

(c) find linearly independent **eigenvectors** associated with the eigenvalues. (3 scores)

(d) find three linearly independent solutions for the system $X' = AX$. (6 scores)
(you must show that they are linearly independent, 3 scores)

(e) form a fundamental matrix Ω for the system $X' = AX$. (3 scores)

(f) write the general solution of the system $X' = AX$. (2 scores)

(g) the initial value problem has a unique solution, why ? (2 scores)

(h) find the unique solution satisfying the initial conditions. (6 scores)

a) $A=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & 4 & -2 \end{pmatrix}$

b) $|\lambda I_3 - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-6 & 4 \\ 0 & -4 & \lambda+2 \end{vmatrix} = (\lambda-2)(\lambda-6)(\lambda+2)+16(\lambda-2)=0$

➔ $\lambda = 2,\quad 2,\quad 2$

c) $(\lambda I_3 - A)\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -4 & 4 \end{pmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = 0$ ➔ $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

➔corresponding eigenvectors are $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ V_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

d) $\phi_1 = V_1 e^{2t},\ \phi_2 = V_2 e^{2t}$

set $\phi_3 = V_2 t e^{2t} + E_2 e^{2t}$ ➔ $(\lambda I - A)E_2 = V_2$ ➔ $E_2 = \begin{pmatrix} 0 \\ 1 \\ 3/4 \end{pmatrix},\ \phi_3 = \begin{pmatrix} 0 \\ t+1 \\ t+3/4 \end{pmatrix} e^{2t}$

$$\left|\phi_1(0) \quad \phi_2(0) \quad \phi_3(0)\right| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3/4 \end{vmatrix} = -1/4 \neq 0$$

e) $\Omega(t) = [\phi_1 \quad \phi_2 \quad \phi_3] = e^{2t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t+1 \\ 0 & 1 & t+3/4 \end{pmatrix}$

f) $X(t) = \Omega(t)C = e^{2t}\begin{pmatrix} c_1 \\ c_2 + (t+1)c_3 \\ c_2 + (t+3/4)c_3 \end{pmatrix}$

g) because each element of A is continuous (for all real t)

h) $c_1 = 1, \quad c_2 = 11, \quad c_3 = -12$

3. $\begin{aligned} x_1' &= 2x_1 - 5x_2 + 2ie^t \\ x_2' &= x_1 - 2x_2 \end{aligned}$

(a) write the matrices A and G of the system $X' = AX + G$ (2 scores)

(b) find the eigenvalues of the matrix A. (2 scores)

(c) find linearly independent eigenvectors associated with the eigenvalues. (2 scores)

(d) solve the general solution of the system by diagonalization. (12 scores)

(e) solve the general solution of the system by variation of parameters. (12 scores)

a) $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}, G = \begin{bmatrix} 2ie^t \\ 0 \end{bmatrix}, X' = AX + G$

b) $|\lambda I_2 - A| = \begin{vmatrix} \lambda - 2 & 5 \\ -1 & \lambda + 2 \end{vmatrix} = (\lambda - 2)(\lambda + 2) + 5 = 0$

➔ $\lambda = \pm i$

c) $\lambda = i$, $\begin{bmatrix} i-2 & 5 \\ -1 & i+2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$ ➔ $V_1 = \begin{Bmatrix} i+2 \\ 1 \end{Bmatrix} or \begin{Bmatrix} 5 \\ 2-i \end{Bmatrix}$

$\lambda = -i$, $\begin{bmatrix} -i-2 & 5 \\ -1 & -i+2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$ ➔ $V_2 = \begin{Bmatrix} -i+2 \\ 1 \end{Bmatrix} or \begin{Bmatrix} 5 \\ 2+i \end{Bmatrix}$

d) $X = PZ, P = \begin{bmatrix} i+2 & -i+2 \\ 1 & 1 \end{bmatrix}, P^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i-2 \\ -1 & i+2 \end{bmatrix}$

➔ $PZ' = APZ + G$ ➔ $Z' = P^{-1}APZ + P^{-1}G = DZ + P^{-1}G, D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

➔ $Z' = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \frac{1}{2i}\begin{bmatrix} 1 & i-2 \\ -1 & i+2 \end{bmatrix}\begin{Bmatrix} 2ie^t \\ 0 \end{Bmatrix}$

$$\Rightarrow \begin{cases} z_1 = iz_1 + e^t \\ z_2 = -iz_2 - e^t \end{cases} \Rightarrow \begin{cases} z_1 = c_1 e^{it} + \dfrac{1}{1-i} e^t = c_1 e^{it} + \dfrac{1+i}{2} e^t \\ z_2 = c_2 e^{-it} - \dfrac{1}{1+i} e^t = c_2 e^{-it} - \dfrac{1-i}{2} e^t \end{cases}$$

e) $\Omega = \begin{bmatrix} V_1 e^{it} & V_2 e^{-it} \end{bmatrix} = \begin{bmatrix} (2+i)e^{it} & (2-i)e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \Rightarrow \Omega^{-1} = \dfrac{1}{2i} \begin{bmatrix} e^{-it} & -(2-i)e^{-it} \\ -e^{it} & (2+i)e^{it} \end{bmatrix}$

$$U = \int \Omega^{-1} G dt = \int \begin{Bmatrix} e^{-it+t} \\ -e^{it+t} \end{Bmatrix} dt = \begin{Bmatrix} \dfrac{e^{t-it}}{1-i} \\ -\dfrac{e^{t+it}}{1+i} \end{Bmatrix}$$

$$X = \begin{bmatrix} (2+i)e^{it} & (2-i)e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} C + \begin{bmatrix} (2+i)e^{it} & (2-i)e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{Bmatrix} \dfrac{e^{t-it}}{1-i} \\ -\dfrac{e^{t+it}}{1+i} \end{Bmatrix}$$