海洋大學九十四學年度 第二學期　　☑期中 考試命題紙
□期末

考　試　科　目	開課系級	考試日期	印製份數	命題教師	備　　註
工程數學（二）	河海工程學系 2 年級	6 月 10 日		陳桂鴻	

1. $A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & 0 & 2 \\ 2 & -1 & 4 \end{bmatrix}$

(1) Compute $|A|$. 3%

(2) Find all eigenvalues and corresponding eigenvector. 8%

(Hint: only one linear independent eigenvector)

(3) Find generalized eigenvectors and write the transition matrix P of A. 8%

(4) Find P^{-1} 3%

(5) Find the Jordan canonical form of A by using the similar transform ($P^{-1}AP$). 8%

ANS

(1). $|A| = 1$

(2). $|A - \lambda I| = 0$ ； $\lambda_1 = \lambda_2 = \lambda_3 = 1$ ； $\underset{\sim}{x_1} = \begin{Bmatrix} 1 \\ -1 \\ -1 \end{Bmatrix}$ *or* $\underset{\sim}{x_1} = \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$

(3). $\begin{bmatrix} -2 & 1 & -3 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{Bmatrix} x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ -1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ ； $\begin{bmatrix} -2 & 1 & -3 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{Bmatrix} x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_3 \end{Bmatrix} = \begin{Bmatrix} -3 \\ 0 \\ 2 \end{Bmatrix}$

$$\therefore P = \begin{bmatrix} 1 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(4). $P^{-1}=\begin{bmatrix}-2&0&-3\\-2&1&-3\\-1&0&-1\end{bmatrix}$

(5). $J=P^{-1}AP=\begin{bmatrix}-2&0&-3\\-2&1&-3\\-1&0&-1\end{bmatrix}\begin{bmatrix}-1&1&-3\\1&0&2\\2&-1&4\end{bmatrix}\begin{bmatrix}1&0&-3\\-1&1&0\\-1&0&2\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$

2. Given quadratic form $Q(x)=3x_1^2+2x_1x_2+3x_2^2$.

(a) Write the matrix form $\tilde{x}^T A\tilde{x}$ of $Q(x)$, where $\tilde{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ and A is real symmetric matrix. 3%

(b) Find all eigenvalues and corresponding eigenvectors, and write the transition matrix P of A. 5%

(c) Find P^{-1} (Hint: P is orthogonal matrix) 3%

(d) Diagonalize A by the similar transform ($P^{-1}AP$). 5%

(e) Find A^{100}. 5%

(f) Transform $Q(x)$ to the standard form ($\sum_{i=1}^{2}\lambda_i y_i^2$) by using the principal axis theorem. 6%

(g) What is the definite property of $Q(x)$? (Negative definite or positive definite) 3%

ANS

(a). $Q(x)=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}3&1\\1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$; $\therefore A=\begin{bmatrix}3&1\\1&3\end{bmatrix}$

(b). $|A-\lambda I|=0$; $\lambda_1=2\quad \lambda_2=4$; $\underset{\sim}{x_1}=\begin{Bmatrix}1\\-1\end{Bmatrix}\quad \underset{\sim}{x_2}=\begin{Bmatrix}1\\1\end{Bmatrix}$; $\therefore P=\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$

(c). $\because P$ is unit prthogonal matrix $\therefore P^{-1}=P^T=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$

(d). $D_A = P^{-1}AP = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

(e). $A^{100} = PD_A{}^{100}P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 4^{100} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2^{100}+4^{100} & -2^{100}+4^{100} \\ -2^{100}+4^{100} & 2^{100}+4^{100} \end{bmatrix}$

(f).令 $\underset{\sim}{x} = P\underset{\sim}{y} \quad \therefore Q(x) = P^{-1}AP\underset{\sim}{y} = 2y_1{}^2 + 4y_2{}^2$

(g).$\because \lambda_1, \lambda_2 > 0$；$\therefore$ 為恆正

Mid-term Exam 3[th]

10 June 2005

1) $A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

a) is the matrix A symmetric ? (2 scores)

b)find the eigenvalues of the matrix (hint: one of the eigenvalues is -2, you guess or solve the others) (4 scores)

c) find three linearly independent eigenvectors associated with the eigenvalues (12 scores) (you must show that they are linearly independent, though they are not unique)

d) find a matrix P such that $P^{-1}AP$ is a diagonal matrix (6 scores) (you must show the result of $P^{-1}AP$)

e) find a matrix Q such that $Q^{-1}A^5Q$ is a diagonal matrix (6 scores) (you must show the result of $Q^{-1}A^5Q$)

a) the matrix A is symmetric ($A^t = A$)

b) $P_A(\lambda) = \begin{vmatrix} \lambda & 1 & -1 \\ 1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda + 2$

find the eigenvalues of A ➔ $P_A(\lambda) = 0 \Longrightarrow \lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = -2$

c)find **an** eigenvector associated with $\lambda_1 = 1, \quad \lambda_2 = 1$

$$➔ (1 \cdot I_3 - A)X = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O \; ➔ \; x_1 + x_2 - x_3 = 0$$

$$➔ X = \begin{Bmatrix} \alpha \\ \beta \\ \alpha + \beta \end{Bmatrix} = \alpha \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} + \beta \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} ➔ \textit{choose} \;\; V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \;\; V_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

find **an** eigenvector associated with $\lambda_3 = -2$

$$➔ (-2 \cdot I_3 - A)X = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O \; ➔ \; \begin{matrix} -2x_1 + x_2 - x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ -x_1 - x_2 - 2x_3 = 0 \end{matrix}$$

$$➔ X = \begin{Bmatrix} \alpha \\ \alpha \\ -\alpha \end{Bmatrix} = \alpha \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} ➔ \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$c_1V_1 + c_2V_2 + c_3V_3 = c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}1\\1\\-1\end{pmatrix} = 0 \Rightarrow \begin{matrix} c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \\ c_1 + c_2 - c_3 = 0 \end{matrix} \Rightarrow \begin{vmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & -1\end{vmatrix} \neq 0, \Rightarrow V_1, \quad V_2, \quad V_3$$ are

linearly independent (the equality holds only if $c_1 = c_2 = c_3 = 0$).

d) $\Rightarrow P = (V_1 \quad V_2 \quad V_3) \Rightarrow P^{-1}AP = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix}$

e) $\Rightarrow Q = (V_1 \quad V_2 \quad V_3)$

$$Q^{-1}AQQ^{-1}AQQ^{-1}AQQ^{-1}AQQ^{-1}AQ$$

$$= Q^{-1}A^5Q = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix}^5 = \begin{pmatrix}1^5 & 0 & 0\\0 & 1^5 & 0\\0 & 0 & (-2)^5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -32\end{pmatrix}$$

2)

$$A = \begin{pmatrix}0 & -1 & 0\\1 & 0 & -1-i\\0 & 1-i & 0\end{pmatrix}$$

a) find $\overline{A}$ (the conjugate of A) (3 scores)

b) find A^t (the transpose of A) (3 scores)

c) is the matrix A hermitian or skew-hermitian ? (3 scores)

d) find the eigenvalues of the matrix A (9 scores)

e) find the eigenvectors associated with the eigenvalues (9 scores)

f) find a matrix Z to diagonalize the matrix A (3 scores)

a) $\overline{A} = \begin{pmatrix}0 & -1 & 0\\1 & 0 & -1+i\\0 & 1+i & 0\end{pmatrix}$

b) $A^t = \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 1-i\\0 & -1-i & 0\end{pmatrix}$

c) $\overline{A} = -A^t \Rightarrow$ the matrix A is skew-hermitian

d) $P_A(\lambda) = \lambda(\lambda^2 + 3)$, The eigenvalues of A are $\lambda = 0, \quad \sqrt{3}i, \quad -\sqrt{3}i$

e)Find **an** eigenvector associated with $\lambda_1 = 0$

$$\Rightarrow (0\cdot I_3 - A)X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1+i \\ 0 & -1+i & 0 \end{pmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O \Rightarrow \begin{matrix} x_2 = 0 \\ -x_1 + (1+i)x_3 = 0 \\ (-1+i)x_2 = 0 \end{matrix}$$

$$\Rightarrow x_1 = \alpha(1+i),\ \ x_2 = 0,\ \ x_3 = \alpha$$

$$\Rightarrow V_1 = \begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix}$$

Find **an** eigenvector associated with $\lambda_2 = \sqrt{3}i$

$$\Rightarrow (\sqrt{3}i\cdot I_3 - A)X = \begin{pmatrix} \sqrt{3}i & 1 & 0 \\ -1 & \sqrt{3}i & 1+i \\ 0 & -1+i & \sqrt{3}i \end{pmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O \Rightarrow \begin{matrix} \sqrt{3}ix_1 + x_2 = 0 \\ -x_1 + \sqrt{3}ix_2 + (1+i)x_3 = 0 \\ (-1+i)x_2 + \sqrt{3}ix_3 = 0 \end{matrix}$$

$$\Rightarrow x_1 = \alpha,\ \ x_2 = -\sqrt{3}i\alpha,\ \ x_3 = \alpha(-1+i)$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ -\sqrt{3}i \\ -1+i \end{pmatrix}$$

Find **an** eigenvector associated with $\lambda_2 = -\sqrt{3}i$

$$\Rightarrow (-\sqrt{3}i\cdot I_3 - A)X = \begin{pmatrix} -\sqrt{3}i & 1 & 0 \\ -1 & -\sqrt{3}i & 1+i \\ 0 & -1+i & -\sqrt{3}i \end{pmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O \Rightarrow \begin{matrix} -\sqrt{3}ix_1 + x_2 = 0 \\ -x_1 - \sqrt{3}ix_2 + (1+i)x_3 = 0 \\ (-1+i)x_2 - \sqrt{3}ix_3 = 0 \end{matrix}$$

$$\Rightarrow x_1 = \alpha,\ \ x_2 = \sqrt{3}i\alpha,\ \ x_3 = \alpha(-1+i)$$

$$\Rightarrow V_3 = \begin{pmatrix} 1 \\ \sqrt{3}i \\ -1+i \end{pmatrix}$$

f) $\Rightarrow Z = \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} \Rightarrow Z^{-1}AZ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3}i & 0 \\ 0 & 0 & -\sqrt{3}i \end{pmatrix}$