W 海洋大學河海工程學系九十四學年度 第二學期 ■期中 □小考 考試命題紙

考 試 科 目	開課系級	考試日期	印製份數	答案紙	命題教師	備 註
工程數學二	二 A, B	4 月 14 日	111	■ 需 □不需	陳桂鴻 呂學育	第一次大考

1. $\vec{F}=\dfrac{-y}{x^2+y^2}\vec{i}+\dfrac{x}{x^2+y^2}\vec{j}$; Evaluate $\oint_C \vec{F}\cdot dr$

(a)C is shown as Fig1(a). (Hint: Using direct integral) (7%)

(b)C is shown as Fig1(b). (Hint: Using Green's theorem) (7%)

(c) C is shown as Fig1(c). (Hint: Using Green's theorem) (6%)

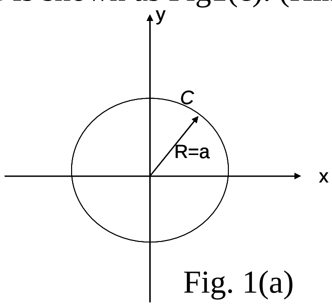

Fig. 1(a)

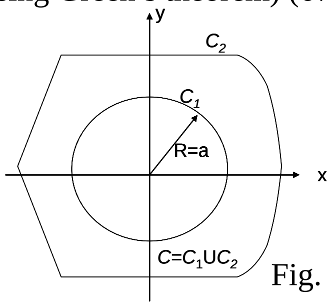

Fig. 1(b)

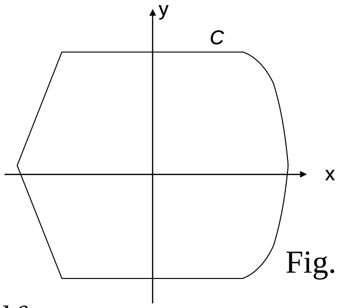

Fig. 1(c)

Ans: (a) Let $x=a\cos\theta$, $dx=-a\sin\theta d\theta$, $y=a\sin\theta$, $dy=a\cos\theta d\theta$

$$\vec{F}=\frac{-\sin\theta}{a}\vec{i}+\frac{\cos\theta}{a}\vec{j},\quad \oint_C \vec{F}\cdot dr=\int_0^{2\pi}(\sin^2\theta+\cos^2\theta)d\theta=2\pi$$

(b) $P=\dfrac{-y}{x^2+y^2}$, $P_y=\dfrac{-x^2+y^2}{(x^2+y^2)^2}$, $Q=\dfrac{x}{x^2+y^2}$, $Q_x=\dfrac{y^2-x^2}{(x^2+y^2)^2}$

$$\oint_C \vec{F}\cdot dr=\int\int_R (Q_x-P_y)dxdy=0$$

(c) $\oint_{c_1}+\oint_{c_2}=0$, $\therefore \oint_{c_1}=-\oint_{c_2}=\oint_{c_2^{-1}}\dfrac{-y}{x^2+y^2}dx+\dfrac{x}{x^2+y^2}dy=2\pi$

2. $\vec{F}=y^3\vec{i}-x^3\vec{j}+z^3\vec{k}$; C is the trace of the cylinder $x^2+y^2=1$ in the plane $x+y+z=1$.

(a) Show that the force is conservative or nonconservative. (5%)

(b) Use Stokes's theorem to evaluate $\oint_C \vec{F}\cdot dr$. (15%)

Ans: (a) $\nabla\times\vec{F}=(-3x^2-3y^2)\vec{k}$, $\therefore \vec{F}$ is nonconservative.

(b) $\phi=x+y+x+1=0$, $\nabla\phi=\vec{i}+\vec{j}+\vec{z}$, $\vec{n}=\dfrac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{z})$, $ds=\sqrt{3}dA$

$$\oint_C \vec{F}\cdot dr=\int\int(\nabla\times\vec{F})\cdot\vec{n}ds=\int\int-3(x^2+y^2)dA=-3\int_0^{2\pi}\int_0^1 r^2 rdrd\theta=\frac{-3}{2}\pi$$

3. The given vector field $\vec{F}(x,y,z)=(x\vec{i}+y\vec{j}+z\vec{k})/(x^2+y^2+z^2)$, S is the region bounded by the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

(a). Find $\nabla\cdot\vec{F},\nabla\times\vec{F}$. (5%)

(b). Find the normal vector $\vec{n}$ of S. (5%)

(c). Use the divergence theorem to find the outward flux $\iint_S(\vec{F}\cdot\vec{n})dS$ of $\vec{F}$. (10%)

Ans: (a) $\nabla\cdot\vec{F}=\dfrac{1}{x^2+y^2+z^2}$, $\nabla\times\vec{F}=0$

(b) $\phi=x^2/a^2+y^2/b^2+z^2/c^2-1$, $\nabla\phi=\dfrac{2x}{a^2}\vec{i}+\dfrac{2y}{b^2}\vec{j}+\dfrac{2z}{c^2}\vec{k}$, $|\nabla\phi|=2\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}}$

$$\vec{n}=\frac{\nabla\phi}{|\nabla\phi|}=(\frac{x}{a^2}\vec{i}+\frac{y}{b^2}\vec{j}+\frac{z}{c^2}\vec{k})/(\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}})$$

(c) $\iint_S F\cdot ndS=\iiint_D divFdV=\int_0^{2\pi}\int_0^{\pi}\int_a^b\dfrac{1}{\rho^2}\rho^2\sin\phi d\rho d\phi d\theta=4\pi(b-a)$

4. Suppose $\vec{r}(t)=t^2\vec{i}+(t^3-2t)\vec{j}+(t^2-5t)\vec{k}$ is the position vector of a moving particle. What are its speed, velocity, acceleration, curvature and tangent line at the point (0,0,0) ? (15 scores)

Ans: $\vec{v}(t)=2t\vec{i}+(3t^2-2)\vec{j}+(2t-5)\vec{k}$, $\vec{a}(t)=2\vec{i}+6t\vec{j}+2\vec{k}$

speed: $|\vec{v}(0)|=\sqrt{(-2)^2+(-5)^2}=\sqrt{29}$

$\vec{v}(0)=-2\vec{j}-5\vec{k}$, $\vec{a}(0)=2\vec{i}+2\vec{k}$

$$\kappa=\frac{|\vec{v}(0)\times\vec{a}(0)|}{|\vec{v}(0)|^3}=\frac{|-4\vec{i}-10\vec{j}+4\vec{k}|}{\sqrt{29}^3}=\frac{2}{29}\sqrt{\frac{33}{29}}$$

tangent line through $(0,0,0)$: $x=0+0t,\ y=0+(-2)t,\ z=0+(-5)t$

5. If S is the portion of the plane $x+2y+3z=6$ in the first octant. For $\vec{F}=y\vec{i}+z\vec{j}+x\vec{k}$

(1) find the area of S (5 scores)

(2) find the upper unit normal of S (5 scores)

(3) use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$,where the curve C is the boundary of S and C is oriented counterclockwise as viewed from above. (10 scores)

Ans: (1) To project S on the xy-plane, use $f(x,y)=z=\dfrac{6-x-2y}{3}$. Then we have

$f_x(x,y)=-\dfrac{1}{3}$ and $f_y(x,y)=-\dfrac{2}{3}$, $dS=\sqrt{1+f_x^2+f_y^2}dA=\sqrt{\dfrac{14}{9}}dA=\dfrac{\sqrt{14}}{3}dA$ Then

$$A=\int_0^6\int_0^{\frac{6-x}{2}}\sqrt{\frac{14}{9}}dydx=\sqrt{\frac{14}{9}}\int_0^6\frac{6-x}{2}dx=\frac{1}{2}\sqrt{\frac{14}{9}}(6x-\frac{1}{2}x^2)\Big|_0^6=3\sqrt{14}$$

(2) If $g(x,y,z)=x+2y+3z=6$, $\vec{n}=\dfrac{\nabla g}{|\nabla g|}=\dfrac{1\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}}$

(3) $\oint_C \vec{F}\cdot d\vec{r}=\iint_S(\nabla\times\vec{F})\cdot\vec{n}dS=\iint_S(-1\vec{i}-1\vec{j}-1\vec{k})\cdot(\dfrac{1\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}})dS=-\dfrac{6}{\sqrt{14}}\iint_R\dfrac{\sqrt{14}}{3}dA=-18$

6. If S is the surface of the region bounded by $x^2+y^2=4$, $z=\sqrt{16-x^2-y^2}$, $z=0$.

$\vec{F}=-y^3\vec{i}-x^3\vec{j}+z^3\vec{k}$

(1) find the volume of the solid bounded by $x^2 + y^2 = 4$, $z = \sqrt{16 - x^2 - y^2}$, $z = 0$. (10 scores)

(2) use the divergence theorem to find the outward flux $\iint_S (\vec{F} \cdot \vec{n})\, dS$ (15 scores)

Ans: (1) $V = \iiint_D dV = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{16-x^2-y^2}} rdzdrd\theta = \int_0^{2\pi} \int_0^2 r\sqrt{16-r^2}\, drd\theta$

$$= \int_0^{2\pi} -\frac{1}{3}(16-r^2)^{\frac{3}{2}} \Big|_0^2 = \int_0^{2\pi} -\frac{1}{3}(12\sqrt{12}-64)d\theta = -16\sqrt{3}\pi + \frac{128}{3}\pi$$

(2) $div\,\overline{F} = \frac{\partial(-y^3)}{\partial x} + \frac{\partial(-x^3)}{\partial y} + \frac{\partial(z^3)}{\partial z} = 3z^2$

$$\iint_S \overline{F} \cdot \vec{n} dS = \iiint_D div\,\overline{F} dV = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{16-r^2}} 3z^2 rdzdrd\theta$$

$$= \int_0^{2\pi} \int_0^2 rz^3 \Big|_0^{\sqrt{16-r^2}} drd\theta = \int_0^{2\pi} \int_0^2 r(16-r^2)^{3/2} drd\theta$$

$$= \int_0^{2\pi} -\frac{1}{5}(16-r^2)^{5/2} \Big|_0^2 d\theta = \int_0^{2\pi} -\frac{1}{5}(12^{5/2} - 4^5)d\theta$$

$$= \int_0^{2\pi} \frac{1}{5}(1024 - 144\sqrt{12})d\theta = \frac{2\pi}{5}(1024 - 288\sqrt{3})$$