

考試科目	開課系級	考試日期	印製份數	答案紙	命題教師	備註
工程數學二	二 A, B	5 月 12 日	110	■ 需 □ 不需	陳桂鴻 呂學育	第二次大考

1. $y''(t) + \omega^2 y(t) = F(t)$ where $F(t) = \begin{cases} 1, & t \in (0, \pi) \\ 0, & t \in (\pi, 2\pi) \end{cases}$ and $F(t) = F(t + 2\pi)$. (20%)

(a) Find $y_p(t)$ by using the complex Fourier expansion. (10%)

(b) Plot the frequency spectrum of $y_p(t)$. (5%)

(c) Choice the right answer and explain why when cause the phenomenon of Resonance as (5%)

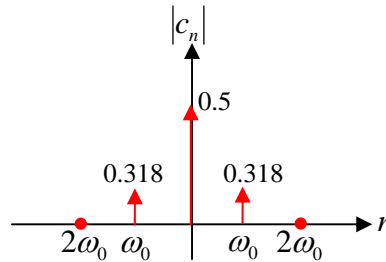
- (1) ω is odd numbers. (2) ω is even numbers.
 (3) ω is integer numbers. (4) The resonance will not occur.

Ans: (a) $F(t) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$; $c_n = \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx = \frac{-1}{2in\pi} [(-1)^n - 1]$

$$y_p(t) = \sum_{n=-\infty}^{\infty} \frac{-1}{2in\pi(-n^2 + \omega^2)} [(-1)^n - 1] e^{inx}$$

(b) $c_0 = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2}$;

n	-2	-1	0	1	2
$ c_n $	0	0.318	0.5	0.318	0



(c) (3) ω is integer numbers.

2. Suppose a uniform beam of length L is simply supported at $x=0$ and at $x=L$. If the load per unit length is given by

$$r(x) = \begin{cases} 0, & 0 < x < L \\ w_0(x-L), & L < x < 2L \\ 0, & 2L < x < 3L \end{cases}, \quad 0 < x < 3L, \quad r(x+3L) = r(x), \quad \text{and then the differential equation for the deflection } y(x) \text{ is}$$

$EI \frac{d^4 y}{dx^4} = r(x)$, where E, I , and w_0 are constants. (40%)

(a) Find the homogenous solution y_h . (5%)

(b) Expand $r(x)$ in a half-range cosine series. (7%)

(c) Find a particular solution $y_p(x)$ by using the Fourier series expansion. (10%)

(d) Expand $r(x)$ in a complex Fourier series and plot frequency spectrum of $r(x)$. (8%)

(e) Find a particular solution $y_p(x)$ by using the complex Fourier series expansion. (10%)

Ans: (a) $y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3$

(b) $r(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi}{3L} x$; $a_0 = \frac{1}{6L} \int_{-3L}^{3L} r(x) dx = \frac{\omega_0 L}{3}$,

$$a_n = \frac{1}{3L} \int_{-3L}^{3L} r(x) \cos \frac{n\pi}{3L} x dx = \frac{2w_0 L}{n\pi} \sin \frac{2n\pi}{3} + \frac{6w_0 L}{(n\pi)^2} (\cos \frac{2n\pi}{3} - \cos \frac{n\pi}{3})$$

(c) $r(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos \frac{2n\pi}{3L} x + b_n \sin \frac{2n\pi}{3L} x$

$$a_0 = \frac{1}{3L} \int_0^{3L} r(x) dx = \frac{w_0 L}{6}; \quad a_n = \frac{2}{3L} \int_0^{3L} r(x) \cos \frac{2n\pi}{3L} dx = \frac{w_0 L}{n\pi} \left(\sin \frac{4n\pi}{3} \right) + \frac{3w_0 L}{(n\pi)^2} \left(\cos \frac{4n\pi}{3} - \cos \frac{2n\pi}{3} \right);$$

$$b_n = \frac{2}{3L} \int_0^{3L} r(x) \sin \frac{2n\pi}{3L} dx = \frac{-w_0 L}{n\pi} \left(\cos \frac{4n\pi}{3} \right) + \frac{3w_0 L}{2(n\pi)^2} \left(\sin \frac{4n\pi}{3} - \sin \frac{2n\pi}{3} \right)$$

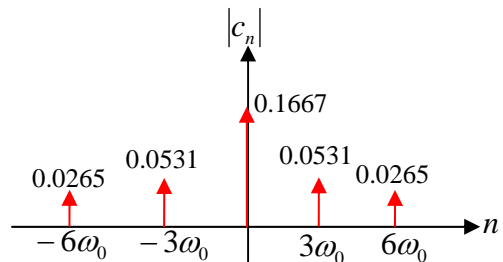
$$\text{Let } y_p(x) = A_0 x^4 + \sum_{n=0}^{\infty} A_n \cos \frac{2n\pi}{3L} x + B_n \sin \frac{2n\pi}{3L} x$$

$$A_0 = \frac{w_0 L}{144EI}, \quad A_n = \left(\frac{3L}{2n\pi} \right)^2 \frac{1}{EI} a_n, \quad B_n = \left(\frac{3L}{2n\pi} \right)^2 \frac{1}{EI} b_n$$

$$(d) r(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2n\pi}{3L} x}; \quad c_n = \frac{1}{3L} \int_{-3L/2}^{3L/2} r(x) e^{-i \frac{2n\pi}{3L} x} dx = \frac{-w_0 L}{2n\pi} i e^{-i \frac{4n\pi}{3}} + \frac{3w_0 L}{(2n\pi)^2} \left(e^{-i \frac{4n\pi}{3}} - e^{-i \frac{2n\pi}{3}} \right)$$

$$c_0 = \frac{1}{3L} \int_L^{2L} w_0 (x-L) dx = \frac{w_0 L}{6}$$

n	-6	-3	0	3	6
$ c_n $	0.0265	0.0531	0.1667	0.0531	0.0265



$$(e) \text{Let } y_p(x) = \sum_{n=-\infty}^{\infty} d_n e^{i \frac{2n\pi}{3L} x}; \quad d_n = \left(\frac{3L}{2n\pi} \right)^4 \frac{1}{EI} c_n$$

3. Consider $f(x) = x + \pi$, $-\pi < x < \pi$

(1) determine whether the function f is even, odd, or neither (3 scores)

ANS neither

(2) find the Fourier series of f on the given interval $(-\pi, \pi)$ (8 scores)

$$\mathbf{ANS} \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) dx = 2\pi$$

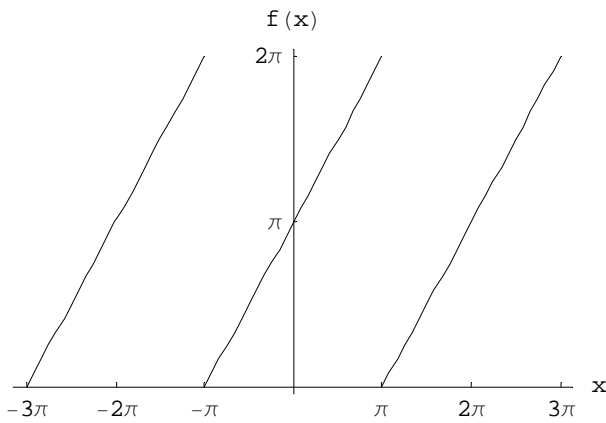
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + \pi) \sin nx dx = \frac{-2}{n} (-1)^n$$

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx$$

(3) give the values that the series will converge at $x = -\pi, 0, \pi/2, \pi$ (4 scores)

ANS



$$\frac{f(-\pi^+) + f(-\pi^-)}{2} = \frac{0 + 2\pi}{2} = \pi$$

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi + \pi}{2} = \pi$$

$$\frac{f(\frac{\pi}{2}^+) + f(\frac{\pi}{2}^-)}{2} = \frac{\frac{3\pi}{2} + \frac{3\pi}{2}}{2} = \frac{3\pi}{2}$$

$$\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{0 + 2\pi}{2} = \pi$$

(4) use the result of (2) to show $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ (5 scores)

ANS Let $x = \frac{\pi}{2} \rightarrow \frac{\pi}{2} + \pi = \pi + \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx \rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx = 2 \cdot 1 + \frac{2}{3} \cdot (-1) + \frac{2}{5} \cdot 1 + \frac{2}{7} \cdot (-1) + \dots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

4. Expand $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq L/2 \\ L-x & \text{for } L/2 < x \leq L \end{cases}$

(1) in a sine series AND give the value that the series will converge at $x = L$ (10 scores)

ANS $a_0 = \frac{2}{L} \int_0^{L/2} x dx + \frac{2}{L} \int_{L/2}^L (L-x) dx = L/2$

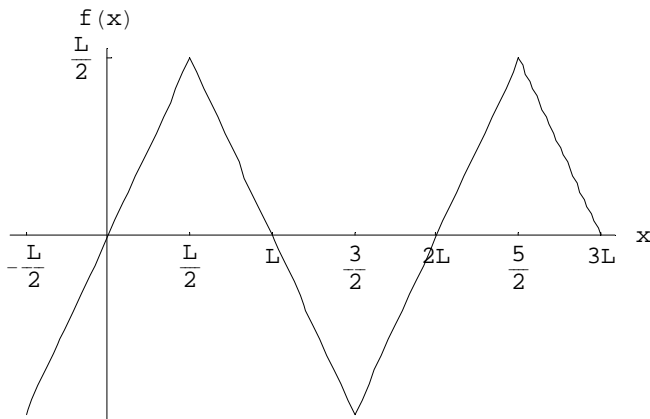
$$a_n = \frac{2}{L} \int_0^{L/2} x \cdot \cos \frac{n\pi x}{L} dx + \frac{2}{L} \int_{L/2}^L (L-x) \cdot \cos \frac{n\pi x}{L} dx = \frac{8L \cos \frac{n\pi}{2} \sin^2 \frac{n\pi}{4}}{n^2 \pi^2}$$

$$= \frac{2L(-1 + 2 \cos \frac{n\pi}{2} - (-1)^n)}{n^2 \pi^2}$$

$$b_n = \frac{2}{L} \int_0^{\frac{L}{2}} x \cdot \sin \frac{n\pi x}{L} dx + \frac{2}{L} \int_{\frac{L}{2}}^L (L-x) \cdot \sin \frac{n\pi x}{L} dx = \frac{16L \cos \frac{n\pi}{4} \sin^3 \frac{n\pi}{4}}{n^2 \pi^2}$$

$$= \frac{4L}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

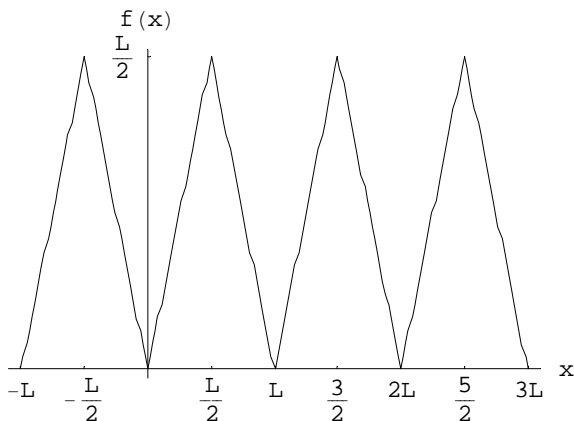
$$f(x) = \sum_{n=1}^{\infty} \frac{4L}{n^2 \pi^2} \sin \frac{n\pi}{2} \cdot \sin \frac{n\pi x}{L}$$



$$\frac{f(L^+) + f(-L^-)}{2} = \frac{0+0}{2} = 0$$

(2) in a cosine series AND give the value that the series will converge at $x = L$
(10 scores)

ANS $f(x) = L/4 + \sum_{n=1}^{\infty} \frac{2L(-1 + 2 \cos \frac{n\pi}{2} - (-1)^n)}{n^2 \pi^2} \cdot \cos \frac{n\pi x}{L} = L/4$



$$\frac{f(L^+) + f(-L^-)}{2} = \frac{0+0}{2} = 0$$

5. $f(x) = \begin{cases} -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$

(1) find the complex Fourier series of f on the given interval (10 scores)

ANS we make use of the following identities due to Euler's formula:

$$e^{in\pi} = e^{-in\pi} = (-1)^n, \quad e^{-2in\pi} = 1, \quad e^{-in\pi/2} = (-i)^n$$

Identifying $p=2$ we have

$$c_n = \frac{1}{4} \int_{-2}^2 f(x) e^{-in\pi x/2} dx = \frac{1}{4} \left[\int_{-2}^0 (-1) e^{-in\pi x/2} dx + \int_0^2 e^{-in\pi x/2} dx \right]$$

$$= \frac{i}{2n\pi} [-1 + e^{in\pi} + e^{-in\pi} - 1] = \frac{i}{2n\pi} [-1 + (-1)^n + (-1)^n - 1] = \frac{1 - (-1)^n}{n\pi i}$$

and

$$c_0 = \frac{1}{4} \int_{-2}^2 f(x) dx = 0$$

Thus

$$f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1 - (-1)^n}{in\pi} e^{in\pi x/2}$$

(2) find the frequency spectrum of the periodic wave that is the periodic extension of the function f (10 scores)

ANS The fundamental period is $T = 4$, so $w = 2\pi/4 = \pi/2$ and the values of nw are $0, \pm\pi/2, \pm\pi, \pm3\pi/2, \dots$. From Problem 1, $c_0 = 0$

and $|c_n| = (1 - (-1)^n)/n\pi$. The table shows some values of n with corresponding

values of $|c_n|$. The graph is a portion of the frequency spectrum.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.1273	0.0000	0.2122	0.0000	0.6366	0.0000	0.6366	0.0000	0.2122	0.0000	0.1273

