

考試科目	開課系級	考試日期	印製份數	答案紙	命題教師	備註
工程數學二	二 A, B	6月2日	110	■ 需 □ 不需	陳桂鴻 呂學育	第三次大考

1. (1) Proof $F[f(t) * g(t)] = F(\omega)G(\omega)$, where $*$ is convolution operator. (10%)

(2) $f(t) = e^{-a|t|}$, $a > 0$, compute $F[f(t)]$. (3%)

(3) $g(t) = \delta(t-1)$, compute $F[g(t)]$. (4%)

(4) Compute $F[f(t) * g(t)] = ?$ (3%)

(5) Solve y_p of $y''(t) + 2y(t) = f(t) * g(t)$. $\left(F^{-1}\left[\frac{1}{2-\omega^2}\right] = \frac{\pi \sin \sqrt{2}t}{\sqrt{2}}$ by using Residue theorem) (10%)

Ans: (1) $F[f(t) * g(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \right] e^{-i\omega t} dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t-\tau)e^{-i\omega(t-\tau)}g(\tau)e^{-i\omega\tau}d\tau \right] dt$
 $= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \bar{f}(t-\tau)\bar{g}(\tau)d\tau \right] dt = \int_{-\infty}^{\infty} \bar{f} * \bar{g} dt = \int_{-\infty}^{\infty} \bar{f}(t)dt \int_{-\infty}^{\infty} \bar{g}(t)dt = \int_{-\infty}^{\infty} \bar{f}(t)e^{-i\omega t} dt \int_{-\infty}^{\infty} \bar{g}(t)e^{-i\omega t} dt$
 $= F(\omega)G(\omega)$

(2) $F[f(t)] = \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t} dt = \frac{2a}{a^2 + \omega^2}$

(3) $F[g'(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = 1$, $F[g(t)] = F[g'(t-1)] = e^{-i\omega}$

(4) $F[f(t) * g(t)] = F(\omega)G(\omega) = \frac{2a}{a^2 + \omega^2} e^{-i\omega}$

(5) $(-\omega^2 + 2)Y(\omega) = \frac{2a}{a^2 + \omega^2} e^{-i\omega}$, $Y(\omega) = \frac{2a}{a^2 + \omega^2} \frac{1}{(-\omega^2 + 2)} e^{-i\omega}$, $Y(\omega) = \left[\frac{2a}{a^2 + \omega^2} + \frac{2a}{(-\omega^2 + 2)} \right] e^{-i\omega}$

$y = F^{-1}\left[\frac{2a}{a^2 + \omega^2} e^{-i\omega}\right] + F^{-1}\left[\frac{2a}{(-\omega^2 + 2)} e^{-i\omega}\right] = \frac{1}{a^2 + 2} e^{-a|t-1|} + \frac{2a}{a^2 + 2} \frac{\pi \sin \sqrt{2}(t-1)}{\sqrt{2}}$

2. (1) Derive the Fourier transform formulation by using the complex Fourier series. (10%)

(2) Write the formulations of transform pairs according to your knowledge. (5%)

(3) Proof the relations of $F(\omega)$ and $A(\omega), B(\omega)$, where $F(\omega)$ is the Fourier transform of $f(t)$, and $A(\omega), B(\omega)$ are the Fourier integral coefficients of $f(t)$. (15%)

Ans: (1) $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n x}$ where $C_n = \frac{1}{T} \int_{-p}^p f(x) e^{-i\omega_n x} dx$

$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-p}^p f(x) e^{-i\omega_n x} dx \right] e^{i\omega_n x} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-p}^p f(x) e^{-i\omega_n x} dx \right] e^{i\omega_n x} \frac{1}{\Delta\omega}$

When $T \rightarrow \infty, x \in (-\infty, \infty)$; $\omega_n = \frac{2n\pi}{T} \rightarrow \omega$; $\Delta\omega = \frac{2\pi}{T} \rightarrow d\omega$; $\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right] e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega, \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

(2)

Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Fourier cosine transform

$$A(\omega) = \int_0^{\infty} f(x) \cos \omega x dx$$

Fourier sine transform

$$B(\omega) = \int_0^{\infty} f(x) \sin \omega x dx$$

Inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

Inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} A(\omega) \cos \omega x d\omega$$

Inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} B(\omega) \sin \omega x d\omega$$

$$(3) F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-\infty}^{\infty} f(x) (\cos \omega x - i \sin \omega x) dx = \int_{-\infty}^{\infty} \bar{A}(\omega) - i \bar{B}(\omega) dx$$

$$\text{where } \bar{A}(\omega) = \int_{-\infty}^{\infty} f(x) \cos \omega x dx, \quad \bar{B}(\omega) = \int_{-\infty}^{\infty} f(x) \sin \omega x dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \omega_n x + b_n \sin \omega_n x], \quad \omega_n = \frac{2n\pi}{T}, \quad a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(x) dx,$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos \omega_n x dx, \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin \omega_n x dx$$

$$\begin{aligned} f(x) &= \frac{1}{2} \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt + \sum_{n=1}^{\infty} \left[\frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \omega_n t dt \cos \omega_n x + \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \omega_n t dt \sin \omega_n x \right] \\ &= \frac{1}{\pi} \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega \end{aligned}$$

$$\text{where } A(\omega) = \int_{-\infty}^{\infty} f(x) \cos \omega x dx, \quad B(\omega) = \int_{-\infty}^{\infty} f(x) \sin \omega x dx$$

$$\bar{A}(\omega) = A(\omega), \quad \bar{B}(\omega) = B(\omega), \quad \text{故得證}$$

3. (1) The Heaviside function $H(t)$ is given by $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

Find the Fourier transform of $f(t) = H(t)e^{-at}$ with a a positive constant. (4 scores)

(2) Determine the value of $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{3i\omega}}{4+i\omega} e^{i\omega t} d\omega$ (5 scores)

(3) Find the Fourier transform of the Dirac delta function $\delta(t)$, $F\{\delta(t)\}$ (4 scores)

(4) Find the Fourier transform of $F\{\delta(t)\}$, $F\{F\{\delta(t)\}\}$ (5 scores)

(5) Find the Fourier transform of $\cos(5t)$, $F\{\cos(5t)\}$ (5 scores)

$$\boxed{\text{ANS}} (1) F\{f(t)\} = \int_{-\infty}^{\infty} H(t) e^{-at} e^{-i\omega t} dt = \int_0^{\infty} e^{-(a+i\omega)t} dt = -\frac{1}{a+i\omega} e^{-(a+i\omega)t} \Big|_0^{\infty} = \frac{1}{a+i\omega}$$

$$(2) \quad F^{-1} \left\{ \frac{1}{4+i\omega} \right\} = f(t) = H(t)e^{-4t} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4+i\omega} e^{i\omega t} d\omega = f(t) = H(t)e^{-4t}$$

Applying the time time-shifting theorem on $f(t+3)$

$$F \{ f(t+3) \} = \frac{e^{3i\omega}}{4+i\omega}$$

Therefore

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{3i\omega}}{4+i\omega} e^{i\omega t} d\omega = f(t+3) = H(t+3)e^{-4(t+3)}$$

$$(3) \quad F \{ \delta(t) \} = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = e^{-i\omega t} \Big|_{t=0} = 1$$

$$(4) \quad F \{ F \{ \delta(t) \} \} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

$$(5) \quad F \{ \cos(5t) \} = F \left\{ \frac{e^{i5t} + e^{-i5t}}{2} \right\} = \frac{1}{2} [2\pi\delta(\omega+5) + 2\pi\delta(\omega-5)] = \pi\delta(\omega+5) + \pi\delta(\omega-5)$$

4. (1) Solve $\frac{d^2 y(t)}{dt^2} + 4y(t) = \cos(\omega_0 t)$ using Fourier transform with ω_0 a real number (15 scores)

(2) Discuss your result obtained in (1) if $\omega_0 = 2$ (5 scores)

$$\boxed{\text{ANS}} (1) \quad F \{ \cos(\omega_0 t) \} = \frac{1}{2} [2\pi\delta(\omega + \omega_0) + 2\pi\delta(\omega - \omega_0)] = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

$$(2) \quad \frac{d^2 y(t)}{dt^2} + 4y(t) = \cos(\omega_0 t) \rightarrow F \left\{ \frac{d^2 y(t)}{dt^2} + 4y(t) \right\} = F \{ \cos(\omega_0 t) \}$$

$$\rightarrow (i\omega)^2 F \{ y(t) \} + 4F \{ y(t) \} = F \{ \cos(\omega_0 t) \}$$

$$\rightarrow F \{ y(t) \} = \frac{F \{ \cos(\omega_0 t) \}}{(i\omega)^2 + 4} = \frac{\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)}{4 - \omega^2}$$

$$\begin{aligned} \rightarrow y(t) &= F^{-1} \{ F \{ y(t) \} \} \\ &= F^{-1} \left\{ \frac{\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)}{4 - \omega^2} \right\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)}{4 - \omega^2} e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \left\{ \left[\frac{\pi}{4 - \omega^2} e^{i\omega t} \right] \Big|_{\omega = -\omega_0} + \left[\frac{\pi}{4 - \omega^2} e^{i\omega t} \right] \Big|_{\omega = \omega_0} \right\} \\ &= \frac{1}{2\pi} \left\{ \frac{\pi}{4 - \omega_0^2} e^{-i\omega_0 t} + \frac{\pi}{4 - \omega_0^2} e^{i\omega_0 t} \right\} \\ &= \frac{1}{2} \frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{4 - \omega_0^2} = \frac{\cos \omega_0 t}{4 - \omega_0^2} \end{aligned}$$

If $\omega_0 = 2$

$$y(t) = \frac{\cos \omega_0 t}{4 - \omega_0^2} \Big|_{\omega_0=2} \rightarrow \infty \quad \rightarrow \text{Resonant}$$

5. (1) Find the Fourier sine transform of $f(t) = e^{-at}$ with a a positive constant. (5 scores)

(2) If we assume that $f(t) \rightarrow 0, f'(t) \rightarrow 0$ as $t \rightarrow \infty$, solve the boundary-value problem $y''(t) - 2y(t) = e^{-t}$, $0 < t < \infty$, with $y(0) = y_0$. (12 scores)

$$\boxed{\text{ANS}} \quad (1) \quad F_s \{ e^{-at} \} = \int_0^{\infty} e^{-at} \sin \omega t \, dt = \frac{\omega}{a^2 + \omega^2} \quad (a > 0)$$

(2) We take the Fourier sine transform of the ODE

$$\rightarrow F_s \{ y''(t) - 2y(t) \} = F_s \{ e^{-t} \} \rightarrow F_s \{ y''(t) \} - 2F_s \{ y(t) \} = F_s \{ e^{-t} \}$$

$$\rightarrow -\omega^2 F_s \{ y(t) \} + \omega y_0 - 2F_s \{ y(t) \} = \frac{\omega}{\omega^2 + 1}$$

$$\rightarrow -F_s \{ y(t) \} (\omega^2 + 2) = \frac{\omega}{\omega^2 + 1} - \omega y_0 \rightarrow F_s \{ y(t) \} = \frac{\omega y_0}{(\omega^2 + 2)} - \frac{\omega}{(\omega^2 + 2)(\omega^2 + 1)}$$

$$\rightarrow y(t) = F^{-1} \left\{ \frac{\omega y_0}{(\omega^2 + 2)} - \frac{\omega}{(\omega^2 + 2)(\omega^2 + 1)} \right\}$$

$$\rightarrow y(t) = F^{-1} \left\{ \frac{\omega y_0}{(\omega^2 + 2)} + \frac{\omega}{(\omega^2 + 2)} - \frac{\omega}{(\omega^2 + 1)} \right\}$$

$$= F^{-1} \left\{ (y_0 + 1) \frac{\omega}{(\omega^2 + 2)} - \frac{\omega}{(\omega^2 + 1)} \right\}$$

$$= (y_0 + 1) F^{-1} \left\{ \frac{\omega}{(\omega^2 + 2)} \right\} - F^{-1} \left\{ \frac{\omega}{(\omega^2 + 1)} \right\}$$

$$= (y_0 + 1) e^{-\sqrt{2}t} - e^{-t}$$

$$\boxed{\frac{1}{(\omega^2 + 2)(\omega^2 + 1)} = \frac{1}{(\omega^2 + 1)} - \frac{1}{(\omega^2 + 2)}}$$