HOMEWORK #4 (Chapter 9 Vector Calculus)

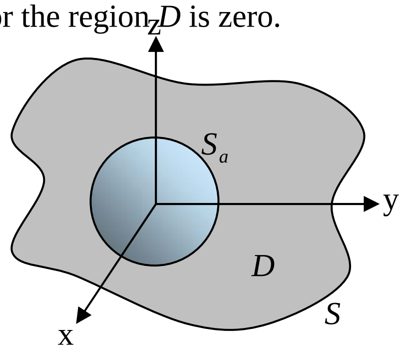
(1). In this problem, use the divergence theorem to find the outward flux $\iint_S (F \cdot n)dS$ of the given vector field *F*. $F(x,y,z) = (x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})/(x^2+y^2+z^2)$; *D* the region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ (Exercises 9.16 problem 9).

Ans: $divF = \dfrac{1}{x^2+y^2+z^2}$

$$\iint_S F \cdot n dS = \iiint_D divF dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{1}{\rho^2}\rho^2 \sin\phi d\rho d\phi d\theta = 4\pi(b-a)$$

(2). The electric field at a point $P(x,y,z)$ due to a point charge *q* located at the origin is given by the inverse square field $E = q\dfrac{r}{\|r\|^3}$, where $r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. (Exercises 9.16 problem 15)

(a). Suppose *S* is a closed surface, S_a is a sphere $x^2+y^2+z^2=a^2$ lying completely within *S*, and *D* is the region bounded between S and S_a. See Figure. Show that the outward flux of E for the region *D* is zero.



Ans:

$$divE = q\left[\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\right] = 0$$

$$\iint_{S \cup S_a} (E \cdot n)dS = \iiint_D divE dV = 0$$

(b). Use the result of part (a) to prove Gauss' law: $\iint_S (E \cdot n) dS = 4\pi q$, that is the outward flux of the electric field E through any closed surface (for which the divergence theorem applies) containing the origin is $4\pi q$.

Ans: $\int\int_S (E \cdot n) dS + \int\int_{S_A} (E \cdot n) dS = 0$ and $\int\int_S (E \cdot n) dS = -\int\int_{S_a} (E \cdot n) dS$

On S_a, $|r| = a$, $n = -(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) / a = -r / a$ and $E \cdot n = -q / a^2$

$\int\int_S (E \cdot n) dS = -\int\int_{S_a} (E \cdot n) dS = 4\pi q$