HOMEWORK #9 (Chapter 15 Integral Transform Method)

In problem 1, find the Fourier integral representation of the given function.

(1) $f(x)=\begin{cases}0, & x<\pi\\ 4, & \pi<x<2\pi\\ 0, & x>2\pi\end{cases}$ (Exercises 15.3 problem 2)

Ans: $A(w)=4\dfrac{\sin 2\pi w-\sin\pi w}{w}$, $B(w)=4\dfrac{\cos\pi w-\cos 2\pi w}{w}$;

$$f(x)=\frac{4}{\pi}\int_0^\infty\frac{\sin w(2\pi-x)-\sin w(\pi-x)}{w}dw$$

In problem 2, represent the given function by an appropriate cosine or sine integral.

(2) $f(x)=\begin{cases}0, & |x|<1\\ \pi, & 1<|x|<2\\ 0, & |x|>2\end{cases}$ (Exercises 15.3 problem 8)

Ans: $A(w)=\pi\int_1^2\cos wx dx=\pi\dfrac{\sin 2w-\sin w}{w}$; $f(x)=2\int_0^\infty\dfrac{\sin 2w-\sin w}{w}dw$

In problem 3, find the cosine and sine integral representations of the given function.

(3) $f(x)=e^{-x}-e^{-3x}$, $x>0$ (Exercises 15.3 problem 14)

Ans: $e^{-kx}=\dfrac{2k}{\pi}\int_0^\infty\dfrac{\cos\alpha x}{1+\alpha^2}d\alpha$ and $e^{-kx}=\dfrac{2}{\pi}\int_0^\infty\dfrac{\alpha\sin\alpha x}{k^2+\alpha^2}d\alpha$

cosine integral representations

$$e^{-x}-e^{-3x}=\frac{4}{\pi}\int_0^\infty\frac{3-\alpha^2}{(1+\alpha^2)(9+\alpha^2)}\cos\alpha x d\alpha$$

sine integral representations

$$e^{-x}-e^{-3x}=\frac{16}{\pi}\int_0^\infty\frac{\alpha\sin\alpha x}{(1+\alpha^2)(9+\alpha^2)}d\alpha$$

(4) Use the complex form (15) to find the Fourier integral representation of $f(x)=e^{-|x|}$. Show that the result is the same as that obtained from (2). (Exercises 15.3 problem 20).

Ans: $C(\alpha)=\int_{-\infty}^\infty e^{-|x|}\cos\alpha x dx+i\int_{-\infty}^\infty e^{-|x|}\sin\alpha x dx$

The imaginary part in the last line is zero since the integrand is an odd function of x.

$$C(\alpha)=\int_{-\infty}^\infty e^{-|x|}\cos\alpha x dx=\int_0^\infty e^{-x}\cos\alpha x dx=\frac{2}{1+\alpha^2}$$

$$f(x)=\frac{1}{\pi}\int_{-\infty}^\infty\frac{\cos\alpha x}{1+\alpha^2}d\alpha=\frac{2}{\pi}\int_0^\infty\frac{\cos\alpha x}{1+\alpha^2}d\alpha$$