Integration and Differentiation of Fourier Series

Consider an infinite series of function

$$f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots = \sum_{n=1}^{\infty} f_n(x)$$

Such a series is said to be convergent for a given value of x if its partial sums

$$s_N(x) = \sum_{n=1}^{N} f_n(x) \qquad (N = 1, 2, ...)$$

have a finite limit $s(x) = \lim_{N\to\infty} S_N(x)$

where $s(x)$ is said to be the sum of the series, and is a function of x. If the series converges for all x in the interval $[a, b]$, then its sum $s(x)$ is defined on the whole interval $[a, b]$.

Weierstrass's M-test

If the series of positive numbers $M_1 + M_2 + \cdots + M_n + \cdots$

converges and if for any x in the interval $[a, b]$ we have $|f_n(x)| \le M_n$ from a certain n on, then $f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly (and absolutely) on $[a, b]$.

Theorem

If the terms of the series are continuous on $[a, b]$ and if the series is uniformly convergent on $[a, b]$, then

a) The sum of the series is continuous;

b) The sum can be integrated term by term, namely

$$\int_a^b \left[\sum_{n=1}^{\infty} f_n(x)\right] dx = \int_a^b s(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$$

Theorem

If the series converges, if the terms are differentiable and if the series

$$f_1'(x) + f_2'(x) + \cdots + f_n'(x) + \cdots = \sum_{n=1}^{\infty} f_n'(x)$$

is uniformly convergent on $[a, b]$, then $\left(\sum_{n=1}^{\infty} f_n(x)\right)' = s'(x) = \sum_{n=1}^{\infty} f_n'(x)$

Namely, the series can be differentiated term by term.