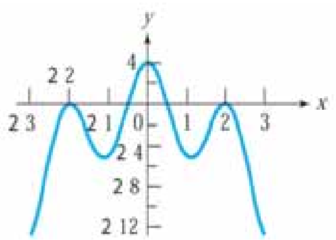
Fourier series of even and odd functions

If $f(x)$ is an even or odd function, then some of the Fourier coefficients can be immediately to be zero, and we need not carry out the integrations explicitly.

Definition:

1) $f(x)$ is an even function on $[-L, \quad L]$ if $f(-x) = f(x)$ for $-L \le x \le L$.

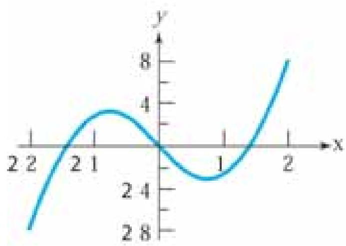
➔the graph of an even function is symmetrical about the y axis.



Graph of a typical even function symmetric about the y axis

2) $f(x)$ is an odd function on $[-L, \quad L]$ if $f(-x) = -f(x)$ for $-L \le x \le L$.

➔ the graph of an odd function is symmetrical about the origin.



Graph of a typical odd function, symmetric through the origin

Then even and the odd functions behave like even and odd integers under multiplication satisfying the following properties:

even • even=even

$h(x) = f(x) \cdot g(x)$

$h(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = h(x)$

odd • odd=even

$$h(x) = f(x) \cdot g(x)$$
$$h(-x) = f(-x) \cdot g(-x) = \left(- f(x)\right) \cdot \left(- g(x)\right) = f(x) \cdot g(x) = h(x)$$

even • odd=odd

$$h(x) = f(x) \cdot g(x)$$
$$h(-x) = f(-x) \cdot g(-x) = f(x) \cdot \left(- g(x)\right) = - f(x) \cdot g(x) = -h(x)$$

The integration of an even/odd function on $[-a, \quad a]$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

If we set $x = -y$ ➔ $dx = -dy$

If $f(x)$ is an even function on $[-a, \quad a]$

➔ $\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-y)(-dy) = -\int_{a}^{0} f(y)dy = \int_{0}^{a} f(y)dy$

➔ $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

If $f(x)$ is an odd function on $[-a, \quad a]$

➔ $\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-y)(-dy) = \int_{a}^{0} f(y)dy = -\int_{0}^{a} f(y)dy$

➔ $\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$

Example: It is clear that $\cos(nx)$ is an even function and $\sin(nx)$ is an odd function. If $f(x)$ is an even function with period 2π, then $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd.

Certainly, you'd better take time to practice Examples 13.4~13.5.