Engineering Mathematics II Hw#3 日河工 2B

23 March 2005

1) Let $f(x) = x\sin(x)$ *for* $-\pi \le x \le \pi$ (Section 13.5 Problems 5.)

(a)Write the Fourier series for $f(x)$ on $[-\pi, \ \pi]$

(b)Show that this series can be differentiated term-by-term and use this fact to obtain the Fourier expansion of $\sin(x) + x\cos(x)$ on $[-\pi, \ \pi]$

(c)Write the Fourier series of $\sin(x) + x\cos(x)$ on $[-\pi, \ \pi]$ by computation of the Fourier coefficients and compare the result with that of (b)

ANS

(a)For this problem, the period $p = 2L = 2\pi$.

Step1: $f(-x) = -x\sin(-x) = x\sin(x) = f(x)$ so $f(x)$ is even.

Step2: Since $f(x)$ is even and $\sin\left(\frac{n\pi x}{\pi}\right) = \sin(nx)$ is odd, $f(x)\sin(nx)$ is odd and hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx) = 0$$

Step3: Since $f(x)$ is even and $\cos(nx)$ is even, $f(x)\cos(nx)$ is even and hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)dx$$

Therefore,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin(x)dx = \frac{2}{\pi}\left\{(x)(-\cos(x))\Big|_0^{\pi} - \int_0^{\pi}(1)(-\cos(x))dx\right\}$$
$$= \frac{2}{\pi}\left\{\pi\cdot(-\cos(-\pi)) + \sin(x)\Big|_0^{\pi}\right\}$$
$$= \frac{2}{\pi}\{\pi\} = 2$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin(x)\cos(x)dx = \frac{2}{\pi}\int_0^{\pi} x\frac{\sin(2x)}{2}dx$$
$$= \frac{2}{\pi}\left\{(x)\left(-\frac{\cos(2x)}{4}\right)\Big|_0^{\pi} - \int_0^{\pi}(1)\left(-\frac{\cos(2x)}{4}\right)dx\right\}$$
$$= \frac{2}{\pi}\left\{\pi\cdot\left(-\frac{\cos(2\pi)}{4}\right) + \frac{\sin(2x)}{8}\Big|_0^{\pi}\right\} = -\frac{1}{2}$$

For $n \geq 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\frac{1}{2}\{\sin(1-n)x + \sin(1+n)x\}dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \{x\sin((1-n)x) + x\sin s((1+n)x)\}dx$$

$$= \frac{1}{\pi}\left\{-\frac{x\cos((1-n)x)}{1-n} + \frac{\sin((1-n)x)}{(1-n)^2}\right\}\Big|_0^{\pi} +$$

$$\frac{1}{\pi}\left\{-\frac{x\cos((1+n)x)}{1+n} + \frac{\sin((1+n)x)}{(1+n)^2}\right\}\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\pi\frac{\cos((1-n)\pi)}{1-n}\right\} + \frac{1}{\pi}\left\{-\pi\frac{\cos((1+n)\pi)}{1+n}\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\frac{\cos(\pi)\cos(n\pi)}{1-n}\right\} + \frac{1}{\pi}\left\{-\pi\frac{\cos(\pi)\cos(n\pi)}{1+n}\right\}$$

$$= \frac{(-1)^n}{1-n} + \frac{(-1)^n}{1+n} = 2\frac{(-1)^{n+1}}{n^2-1}$$

Step4: Therefore, the Fourier series is $1 - \frac{1}{2}\cos(x) + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos(nx)$.

(b) $f(x) = x\sin(x)$ is continuous on $[-\pi, \pi]$ and
$f'(x) = \sin(x) + x\cos(x)$, $f''(x) = \cos(x) + \cos(x) - x\sin(x)$ are all continuous on $[-\pi, \pi]$, and $f(\pi) = f(-\pi)$. So the Theorem 13.6 gives us

$$f'(x) = \sin(x) + x\cos(x) = \frac{1}{2}\sin(x) + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}n\sin(nx) \quad \text{for } (-\pi, \pi).$$

(c)Step1: $g(x) = \sin(x) + x\cos(x)$ on $[-\pi, \pi]$
$g(-x) = \sin(-x) - x\cos(-x) = -\sin(x) - x\cos(x) = -g(x)$, so $g(x)$ is odd.

Step2: Similar to what we done in a), but with $a_n = 0$

➔We can get the same Fourier series as term by term differentiation in b)

2) Let $f(x)=x$ for $0\le x<2$ and $f(x+2)=f(x)$ for all x (Section 13.6 Problem 5.)

Find the phase angle form of the Fourier series of the function. Plot some points of the amplitude spectrum of the function. (hint: please refer to Example 13.28)

ANS

For this problem, the period $p=2L=2$.

$$a_0=\int_{-1}^{1} f(x)\ dx=\int_0^2 x\ dx=2$$

$$a_n=\int_0^2 x\cos(n\pi x)\ dx=\left\{x\frac{\sin(n\pi x)}{n\pi}+\frac{\cos(n\pi x)}{(n\pi)^2}\right\}\Big|_0^2=0$$

$$b_n=\int_0^2 x\sin(n\pi x)\ dx=\left\{-x\frac{\cos(n\pi x)}{n\pi}+\frac{\sin(n\pi x)}{(n\pi)^2}\right\}\Big|_0^2=-\frac{2}{n\pi}$$

➔The corresponding Fourier series is $1-\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin(n\pi x)$

$$c_n=\sqrt{a_n^2+b_n^2}=-\frac{2}{n\pi}$$

$$\delta_n=\tan^{-1}(-b_n/a_n)=\tan^{-1}(-\infty)=-\frac{\pi}{2}$$

➔The phase angle form of the Fourier series is $1-\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\cos(n\pi x-\dfrac{\pi}{2})$

The amplitude spectrum, e.g. **Figure 13.37** in the textbook, of the function consists of points $(n\omega_0,\ \ c_n/2)$ with $\omega_0=2\pi/p=\pi$.

3) Let f has period 3 and $f(x)=2x$ for $0\le x<3$ (Section 13.7 Problem 1.)

(a)Write the complex Fourier series of f

(b)Determine what this series converges to

(c)Plot some points of the frequency spectrum

(hint: please refer to Example 13.29)

ANS

(a)For this problem, the period $p=3$.

$$d_0=\frac{a_0}{2}=\frac{1}{3}\int_{-3/2}^{3/2} f(x)\ dx=\frac{1}{3}\int_0^3 2x\ dx=3$$

$$d_n=\frac{1}{2}(a_n-ib_n)=\frac{1}{p}\int_{-p/2}^{p/2} f(x)e^{-in\omega_0 x}\ dx=\frac{1}{3}\int_0^3 2xe^{-2n\pi x/3}=\frac{3}{n\pi}i$$

The complex Fourier series of f is $3+\frac{3i}{\pi}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{1}{n}e^{2n\pi ix/3}$

(b)Note that the complex Fourier series of f converges to 3 if $x=0, or \quad x=3$ and converges to $2x \ if \ 0<x<3$

(c)As shown in Figure 13.49 of the textbook, the frequency (or amplitude) spectrum is a plot of points $\left(n\omega_0, \ \left|d_n\right|\right)$ with $\omega_0 = 2\pi/p = 2\pi/3$, $\left|d_0\right| = 3$,

$\left|d_n\right| = \sqrt{(3/n\pi)^2} = \frac{3}{n\pi}$.